151163A - Financial Econometrics

IIa. Stationary Processes: Stationarity and Linear Process

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Stochastic Processes

Definition

A stochastic process $\{X_t, t \in \mathcal{T}\}$ *is a collection, or a family of random variables ordered by the time index* $t \in \mathcal{T}$.

The expected value and the (auto)covariance are given by

$$\begin{aligned}\mathbb{E}\left[X_t\right] &= \mu_t \\ \text{cov}(X_t, X_{t-h}) &= \mathbb{E}\left[(X_t - \mu_t)(X_{t-h} - \mu_{t-h})\right] = \gamma_X(t, h).\end{aligned}$$

In particular, the variance of X_t is

$$\gamma_X(t, 0) = \text{var}(X_t) = \mathbb{E}\left[(X_t - \mu_t)^2\right].$$

The autocorrelation is defined as

$$\rho_X(t, h) = \frac{\text{cov}(X_t, X_{t-h})}{\sqrt{\text{var}(X_t)\,\text{var}(X_{t-h})}} = \frac{\gamma_X(t, h)}{\sqrt{\gamma_X(t, 0)\gamma_X(t-h, 0)}}$$

Main challenges in studying stochastic processes:

- Future observations are not available at time t.
- At every time period, we observe only *one* realization out of the set of all possible outcomes.

Ergodicity

Definition

A stochastic process is ergodic if its time average is the same as its average over the probability space.

Definition

A time series $\{X_t\}$ *is said to be strictly stationary if the joint distribution of* $(X_{t_1}, \ldots, X_{t_k})$ *is identical to that of* $(X_{t_1+t}, \ldots, X_{t_k+t})$ *for all positive integers* t *and* k*.*

Definition

A time series $\{X_t\}$ is said to be weakly stationary if both the mean of X_t and the covariance between X_t and X_{t-h} are finite and time-invariant for all integers h, i.e.,

$$\mathbb{E}\left[X_t\right] = \mu$$
$$\operatorname{cov}(X_t, X_{t-h}) = \gamma_X(h)$$

Moreover, we can write the autocorrelation as

$$\rho_X(h) = \frac{\operatorname{cov}(X_t, X_{t-h})}{\sqrt{\operatorname{var}(X_t)\operatorname{var}(X_{t-h})}} = \frac{\gamma_X(h)}{\gamma_X(0)}.$$

Example

Consider the stochastic process

$$X_t = e_t + e_{t-1}, \qquad e_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1).$$

Find the expected value and autocovariance of X_t. Is X_t weakly stationary? Is X_t strictly stationary?

Linear Process

A time series X_t is said to be a linear process if it has finite variance and can be written as

$$X_t = \mu_X + \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \qquad \varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2) \tag{1}$$

where ε_t is called the *shock* or *innovation* at time t.

Since ε_t is iid with mean 0,

- the mean is

$$\mathbb{E}\left[X_t\right] = \mu_X$$

- the variance is

$$\operatorname{var}(X_t) = \gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} b_j^2$$

If X_t is weakly stationary, then $\operatorname{var}(X_t) < \infty$ and $\{b_j^2\}$ must be a convergent sequence, i.e., $b_j^2 \to 0$ as $j \to \infty$. In other words, the impact of the shock ε_{t-h} on X_t vanishes as j increases.

Since ε_t is iid with mean 0,

- the autocovariance is

$$\gamma_X(h) = \text{cov}(X_t, X_{t-h}) = \sigma^2 \sum_{j=0}^{\infty} b_j b_{j+h}$$

- the autocorrelation is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \frac{\sum_{j=0}^{\infty} b_j b_{j+h}}{\sum_{j=0}^{\infty} b_j^2}$$

If X_t is weakly stationary, then $b_j \to 0$ as $j \to \infty$ and $\rho_X(h) \to 0$ as $h \to \infty$ for a stationary process.

White noise sequence

Definition

A time-series X_t is called a white noise if $\{X_t\}$ is a sequence of independent and identically distributed (iid) random variables with finite mean μ_X and variance σ_X^2.

White noise sequence

If $X_t \overset{\text{iid}}{\sim} (\mu_X, \sigma_X^2)$, then

- $\mathbb{E}[X_t] = \mu_X$,
- $\text{var}(X_t) = \sigma_X^2$,
- $\gamma_X(h) = 0$ for all $h > 0$,
- $\rho_X(h) = 0$ for all $h > 0$.

Therefore, X_t is weakly stationary.

Autocorrelation

The lag h sample autocorrelation of X_t is defined as

$$\widehat{\rho}_X(h) = \frac{\sum_{t=h+1}^{T}(X_t - \overline{X})(X_{t-h} - \overline{X})}{\sum_{t=1}^{T}(X_t - \overline{X})^2}$$

where $\overline{X} = T^{-1}\sum_{t=1}^{T} X_t$ is the sample mean of X_t. If X_t is an iid sequence with finite variance, then

$$\widehat{\rho}_X(h) \xrightarrow{d} \mathcal{N}(0, T^{-1}), \qquad h = 1, 2, \ldots.$$

We can therefore test whether X_t is iid by the t-test.

Alternatively, we can test the joint hypothesis

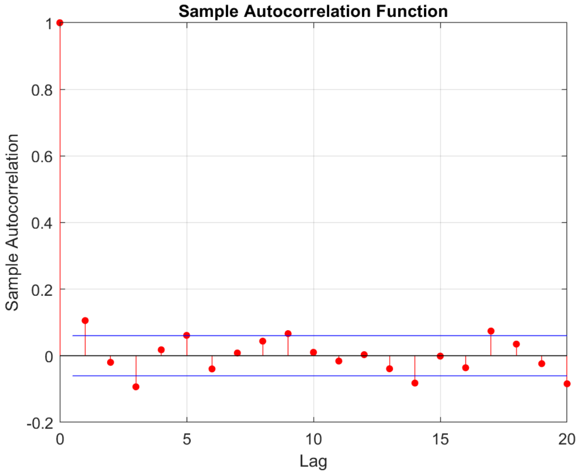
$$\begin{cases} H_0 : \rho_X(1) = \cdots = \rho_X(m) = 0 \\ H_1 : \rho_X(i) \neq 0 \text{ for some } i \in \{1, \ldots, m\} \end{cases}$$

with the statistic

$$Q(m) = T(T+2) \sum_{j=1}^{m} \frac{\widehat{\rho}_X(j)^2}{T-j}$$

Under H_0, $Q(m) \xrightarrow{d} \chi^2_m$.

Autocorrelation of US market returns



m	$Q(m)$	$\chi^2_m(95\%)$	$\chi^2_m(99\%)$	Reject H_0?
1	12.30	3.84	6.63	✓
2	12.74	5.99	9.21	✓
3	22.36	7.81	11.34	✓
4	22.71	9.49	13.28	✓
5	26.85	11.07	15.09	✓
6	28.59	12.59	16.81	✓
7	28.66	14.07	18.48	✓
8	30.80	15.51	20.09	✓
9	35.70	16.92	21.67	✓
10	35.81	18.31	23.21	✓