



151163A - Financial Econometrics

IIb. Stationary Processes:
Autoregressive (AR) Model

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An AR(1) model reads

$$X_t = a_0 + a_1 X_{t-1} + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} (0, \sigma^2). \quad (1)$$

Conditional on X_{t-1} ,

$$\mathbb{E}[X_t | X_{t-1}] = a_0 + a_1 X_{t-1}, \quad \text{var}(X_t | X_{t-1}) = \sigma^2.$$

By repeated substitutions of Eq.(1),

$$\begin{aligned}X_t &= a_0 + a_1 X_{t-1} + \varepsilon_t \\&= a_0 + a_1 [a_0 + a_1 X_{t-2} + \varepsilon_{t-1}] + \varepsilon_t \\&= (1 + a_1) a_0 + a_1^2 X_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t \\&= \dots \\&= \lim_{J \rightarrow \infty} a_1^J X_{t-J} + a_0 \sum_{j=0}^{\infty} a_1^j + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}.\end{aligned}$$

Therefore, an AR(1) process is also a linear process if the summation converges and if its variance is finite.

If $|a_1| < 1$,

► the mean is

$$\mathbb{E}[X_t] = a_0 \sum_{j=0}^{\infty} a_1^j = \frac{a_0}{1 - a_1}$$

► the variance is

$$\gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} a_1^{2j} = \frac{\sigma^2}{1 - a_1^2}$$

If $|a_1| < 1$,

- ▶ the autocovariance is

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} a_1^j \cdot a_1^{j+h} = \frac{a_1^h \sigma^2}{1 - a_1^2}$$

- ▶ the autocorrelation is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = a_1^h.$$

The above results imply that

1. μ_X exists if $a_1 \neq 1$.
2. $\mu_X = 0$ if and only if $a_0 = 0$.
3. $\gamma_X(h)$ exists if $|a_1| < 1$.
4. If μ_X and $\gamma_X(h)$ exist, then X_t is weakly stationary.

Therefore, an AR(1) process is weakly stationary *if and only if* $-1 < a_1 < 1$, or $|a_1| < 1$. We can write an AR(1) process as

$$X_t = \mu_X + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}.$$

Using the property of the weak stationarity of X_t , we can find its moments more easily. Taking expectation on both sides of Eq.(1),

$$\begin{aligned}\mu_X &= \mathbb{E}[X_t] = a_0 + a_1 \mathbb{E}[X_{t-1}] \\ &= a_0 + a_1 \mu_X\end{aligned}$$

Therefore,

$$\mu_X = \frac{a_0}{1 - a_1}.$$

Substitute $a_0 = \mu_X(1 - a_1)$ into Eq.(1),

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \varepsilon_t. \quad (2)$$

Taking the square and then the expectation of Eq.(2),

$$\begin{aligned} \mathbb{E}[(X_t - \mu_X)^2] &= \gamma_X(0) = a_1^2 \gamma_X(0) + \sigma^2 \\ \implies \gamma_X(0) &= \frac{\sigma^2}{1 - a_1^2}. \end{aligned}$$

Multiplying $(X_{t-h} - \mu_X)$ to Eq.(2) and taking expectation,

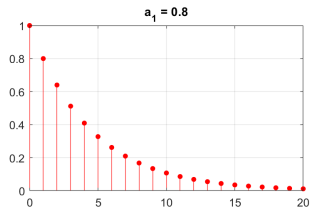
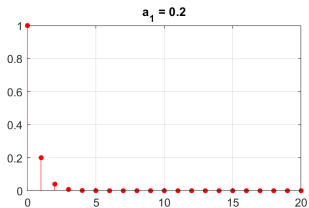
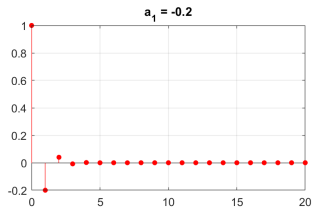
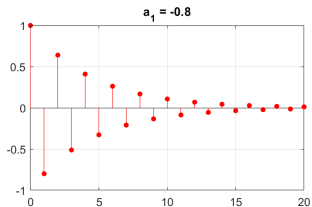
$$\gamma_X(h) = \begin{cases} a_1 \gamma_X(1) + \sigma^2 & \text{if } h = 0 \\ a_1 \gamma_X(h-1) & \text{if } h > 0 \end{cases}$$

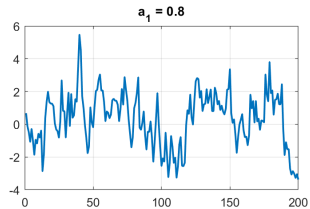
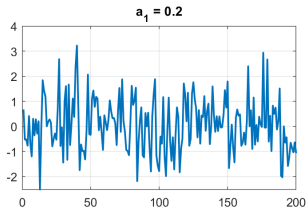
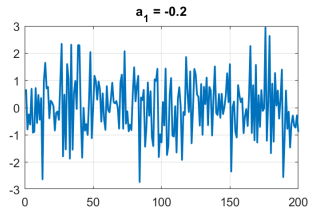
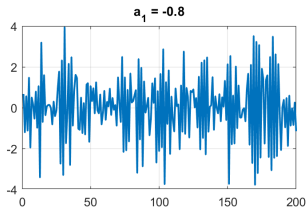
Using the fact that $\gamma_X(h) = \gamma_X(-h)$, we have

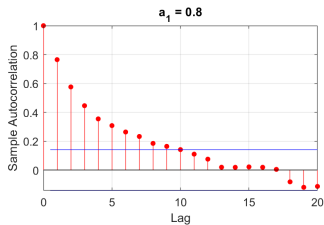
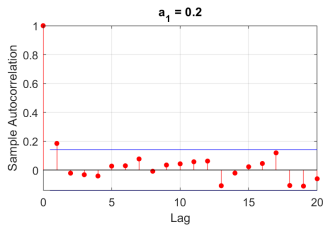
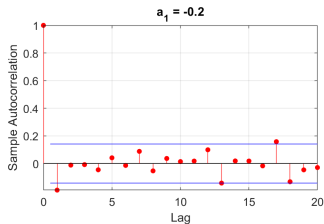
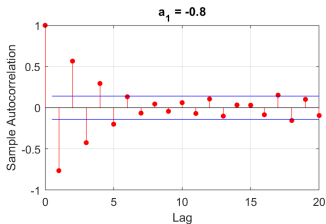
$$\gamma_X(0) = \frac{\sigma^2}{1 - a_1^2}, \quad \gamma_X(h) = a_1 \gamma_X(h-1) = a_1^h \gamma_X(0).$$

Therefore, the autocorrelation function (ACF) of X_t is

$$\rho_X(h) = a_1 \rho_X(h-1) = a_1^h.$$







In general, an AR(p) process is given by

$$X_t = a_0 + a_1 X_{t-1} + \cdots + a_p X_{t-p} + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} (0, \sigma^2).$$

Similarly, if X_t is stationary, its mean can be obtained simply as

$$\mu_X = \mathbb{E}[X_t] = \frac{a_0}{1 - a_1 - \cdots - a_p}.$$

Substituting $a_0 = (1 - a_1 - \cdots - a_p)\mu_X$, we have

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \cdots + a_p(X_{t-p} - \mu_X) + \varepsilon_t.$$

Multiplying $X_{t-h} - \mu_X$ and taking expectation yield

$$\gamma_X(h) - a_1\gamma_X(h-1) - \cdots - a_p\gamma_X(h-p) = \begin{cases} 0 & \text{if } h > 0 \\ \sigma^2 & \text{if } h = 0 \end{cases}$$

Dividing $\gamma_X(0)$ from both equations, and noting that $\gamma_X(-h) = \gamma_X(h)$,

$$\begin{aligned}\sigma^2 &= \gamma_X(0)(1 - a_1\rho_X(1) - \cdots - a_p\rho_X(p)) \\ \rho_X(h) &= a_1\rho_X(h-1) + \cdots + a_p\rho_X(h-p), \quad h > 0\end{aligned}$$

We can then estimate σ^2 and a_1, \dots, a_p from the autocorrelations.

Consider a weakly stationary AR(2) process

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} (0, \sigma^2).$$

Express a_1 and a_2 as a function of the autocorrelations.

Let L be the lag operator, i.e., $LX_t = X_{t-1}$, $L^2X_t = X_{t-2}$, we can write the AR(p) process as

$$(1 - a_1L - \cdots - a_pL^p)(X_t - \mu_X) = \varepsilon_t.$$

Theorem

An AR(p) process X_t is weakly stationary if the solutions of the characteristic equation

$$1 - a_1L - \cdots - a_pL^p = 0$$

are larger than one in absolute value.

Evaluate the stationarity property of the AR(2) process

$$X_t = 0.4X_{t-1} - 0.04X_{t-2} + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} (0, \sigma^2).$$