151163A - Financial Econometrics

IId. Stationary Processes: MA and ARMA Models

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Moving Average Model

Consider the stochastic process

$$X_t = (1 - b_1)a_0 + \varepsilon_t - b_1\varepsilon_{t-1}.$$

Since X_t is just a weighted average of the current and past shocks, it is called a *moving-average* process with order 1, or an MA(1) process.

Re-arranging the MA(1) process, we have

$$\varepsilon_t + a_0 = b_1(\varepsilon_{t-1} + a_0) + X_t = \sum_{j=0}^{\infty} b_1^j X_{t-j}$$

We say X_t is *invertible* if $|b_1| < 1$, so that the realization from the infinite past has little influence on the current shock.

In general, an MA(q) process is given by

$$\begin{aligned} X_t &= \mu_X + \varepsilon_t - b_1\varepsilon_{t-1} - \cdots - b_q\varepsilon_{t-q} \\ &= \mu_X + (1 - b_1L - \cdots - b_qL^q)\varepsilon_t \end{aligned}$$

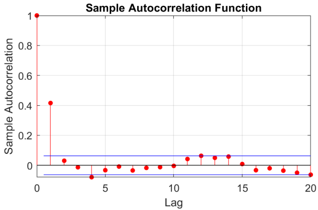
where $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$.

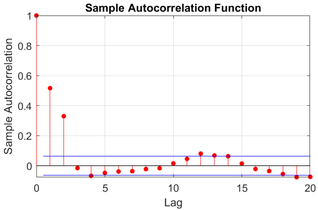
Obviously, by the iid assumption on ε_t,

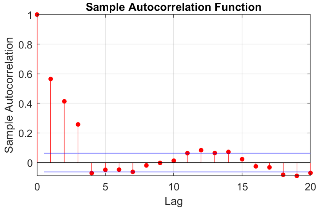
$$\begin{aligned}\mathbb{E}\left[X_t\right] &= \mu_X \\ \operatorname{var}(X_t) &= (1 + b_1^2 + \cdots + b_q^2)\sigma^2 \\ \gamma_X(h) &= \begin{cases}(-b_h + b_{h+1}b_1 + \cdots + b_q b_{q-h})\sigma^2, & h \leq q \\ 0, & h > q\end{cases}\end{aligned}$$

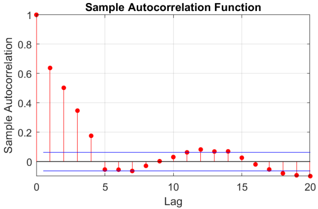
Since they are all time-invariant, an MA(q) process is weakly stationary.

Identifying the order of an MA process









Estimation of an MA process

The estimation of an MA process is typically done by the *maximum-likelihood estimation* (MLE). Consider the MA(1) process

$$X_t = \mu_X + \varepsilon_t - b_1 \varepsilon_{t-1},$$

and assuming that $\varepsilon_0 = 0$, then

$$\begin{aligned}
\varepsilon_1 &= X_1 - \mu_X \\
\varepsilon_2 &= X_2 - \mu_X + b_1 \varepsilon_1 \\
\varepsilon_3 &= X_3 - \mu_X + b_1 \varepsilon_2 \\
&\vdots
\end{aligned}$$

Assuming further that ε_t is normally distributed, the likelihood function is given by

$$\mathcal{L}(\mu_X, b_1, \sigma; X_1, \ldots, X_T) = (2\pi\sigma^2)^{T/2} \exp\left(-\frac{\sum_{t=1}^{T} \varepsilon_t^2}{2\sigma^2}\right)$$

The coefficients can be obtained by maximizing the above likelihood function.

Consider an MA(1) process

$$X_t = \mu_X + \varepsilon_t - b_1 \varepsilon_{t-1}.$$

Then the h-step forecasts are

$$\begin{aligned}
\widehat{X}_t(1) &= \mathbb{E}\left[X_{t+1} | X_t, \ldots\right] = \mu_X - b_1 \varepsilon_t \\
\widehat{X}_t(2) &= \mathbb{E}\left[X_{t+2} | X_t, \ldots\right] = \mu_X
\end{aligned}$$

The process reverts to its mean after two steps.

Forecasting an MA process

Consider an MA(q) process

$$X_t = \mu_X + \varepsilon_t - b_1\varepsilon_{t-1} - \cdots - b_q\varepsilon_{t-q}.$$

Then the h-step forecast is

$$\widehat{X}_t(h) = \mu_X - \begin{cases} 0, & h > q \\ \sum_{j=h}^{q} b_j\varepsilon_{t+h-j}, & h = 1, \ldots, q \end{cases}$$

The variance of the forecast error is

$$\operatorname{var}(X_{t+h} - \widehat{X}_t(h)) = \begin{cases} (1 + b_1^2 + \cdots + b_q^2)\sigma^2, & h > q \\ \sum_{j=0}^{h-1} b_j^2\sigma^2 & h = 1, \ldots, q \end{cases}$$

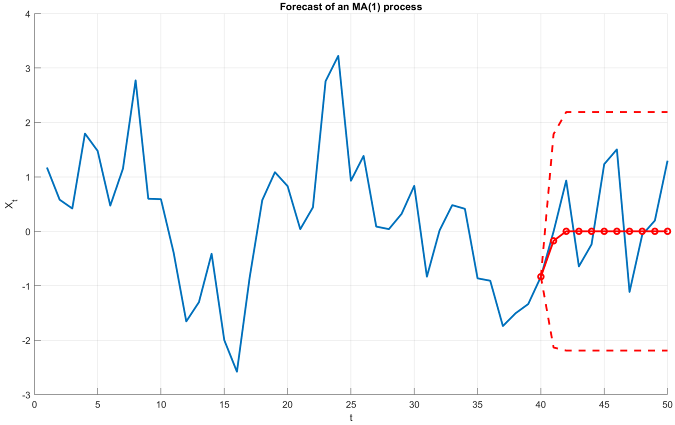
where $b_0 = 1$.

Forecasting an MA process

Procedure of forecasting an MA process:

1. Find the order q of the MA process with the autocorrelation function.
2. Estimate the coefficients $\{b_j\}$ by MLE
3. Obtain the residuals ε_t recursively, assuming the initial shocks $\varepsilon_0, \varepsilon_{-1}, \ldots$.
4. Obtain the forecast $\widehat{X}_t(h)$ by the equation in the last slide.

Forecasting an MA process



ARMA Model

The autoregressive moving-average (ARMA) model combines an AR and MA model. For example, an ARMA(1,1) model is

$$X_t - a_1 X_{t-1} = a_0 + \varepsilon_t - b_1 \varepsilon_{t-1}, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right), \tag{1}$$

where $a_1 \neq b_1$.

If X_t is weakly stationary, then taking expectation of Eq.(1),

$$\mu_X - a_1\mu_X = a_0 \implies \mu_X = \frac{a_0}{1-a_1}.$$

Rewriting Eq.(1) as

$$X_t = a_0 + a_1 X_{t-1} + \varepsilon_t - b_1 \varepsilon_{t-1},$$

and computing the variance of both sides,

$$\begin{aligned}\operatorname{var}(X_t) &= a_1^2 \operatorname{var}(X_{t-1}) + \sigma^2 + b_1^2\sigma^2 - 2a_1b_1\mathbb{E}\left[X_{t-1}\varepsilon_{t-1}\right] \\ &= \frac{(1 - 2a_1b_1 + b_1^2)\sigma^2}{1 - a_1^2}\end{aligned}$$

which requires $|a_1| < 1$.

Properties of an ARMA(1,1) process

To find the autocovariance of X_t, we first write the model as

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \varepsilon_t - b_1\varepsilon_{t-1}.$$

Then we multiply $X_{t-h} - \mu_X$ to the equation and take expectation,

$$\begin{aligned} \gamma_X(1) &= a_1\gamma_X(0) - b_1\sigma^2, & h = 1 \\ \gamma_X(h) &= a_1\gamma_X(h-1), & h > 1. \end{aligned}$$

Similarly, the autocorrelation function (ACF) is given by

$$\rho_X(1) = a_1 - \frac{b_1\sigma^2}{\gamma_X(0)}, \qquad \rho_X(h) = a_1\rho_X(h-1),\ \ h > 1.$$

ARMA(p, q) process

In general, an ARMA(p, q) process is written as

$$X_t = a_0 + a_1 X_{t-1} + \cdots + a_p X_{t-p} + \varepsilon_t - b_1 \varepsilon_{t-1} - \cdots - b_q \varepsilon_{t-q},$$

where $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$. Using the lag polynomial representation,

$$(1 - a_1 L - \cdots - a_p L^p)(X_t - \mu_X) = (1 - b_1 L - \cdots - b_q L^q)\varepsilon_t$$

where

$$\mu_X = \frac{a_0}{1 - a_1 - \cdots - a_p}.$$

The stationarity condition is the same as that of an AR(p) process.

Estimation of an ARMA(p, q) process

Assuming $\varepsilon_j = 0$ for all $j \leq p$, we have

$$\begin{aligned}
\varepsilon_{p+1} &= X_{p+1} - a_1 X_p - \cdots - a_p X_1 - a_0 \\
\varepsilon_{p+2} &= X_{p+2} - a_1 X_{p+1} - \cdots - a_p X_2 - a_0 + b_1 \varepsilon_{p+1} \\
&\vdots
\end{aligned}$$

Assuming again that $\varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma^2\right)$, we can estimate the coefficients by MLE.

Let $\widetilde{\sigma}^2(p,q)$ be the sample variance of the residuals by fitting an ARMA(p,q) process. Then one can select the order by minimizing the criteria functions

$$AIC(p,q) = \ln(\widetilde{\sigma}^2(p,q)) + \frac{2(p+q)}{T}$$
$$BIC(p,q) = \ln(\widetilde{\sigma}^2(p,q)) + \frac{(p+q)\ln T}{T}$$

Representations of an ARMA(p, q) process

Suppose both the AR and MA lag polynomials can be factored such that

$$(1-\phi_1 L)\dots(1-\phi_p L)(X_t-\mu_X) = (1-\theta_1 L)\dots(1-\theta_q L)\varepsilon_t.$$

If $|\phi_j| < 1$ for all j and $\phi_i \neq \theta_j$ for any i, j, we can represent an ARMA process as an MA(∞) process

$$\begin{aligned} X_t &= \mu_X + (1-\phi_1 L)^{-1}\dots(1-\phi_p L)^{-1}(1-\theta_1 L)\dots(1-\theta_q L)\varepsilon_t \\ &= \mu_X + \sum_{j=0}^{\infty} \omega_j \varepsilon_{t-j} \end{aligned}$$

In this case, we say the process is *causal*.

Similarly, if $|\theta_j| < 1$ for all j, then we can represent an ARMA process as an AR(∞) process

$$(1 - \theta_1 L)^{-1} \dots (1 - \theta_q L)^{-1} (1 - \phi_1 L) \dots (1 - \phi_p L)(X_t - \mu_X)$$
$$= \left(1 - \sum_{j=1}^{\infty} \lambda_j L^j\right)(X_t - \mu_X) = \varepsilon_t.$$

In this case, we say the process is *invertible*.

Example

Express the following ARMA(1,1) model as an MA(∞) process.

$$X_t = a_1 X_{t-1} + \varepsilon_t - b_1 \varepsilon_{t-1}, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$