Capital University of Economics and Business

ISEM

Financial Econometrics

AR(1) Process

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Term:
Fall Semester, 2019

October 21, 2019

AR(1) process

We say X_t is an AR(1) process if it takes the form

$$X_t = a_0 + a_1 X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right). \tag{1}$$

AR(1) process as a linear process

Noticing that

$$\begin{aligned}
X_t &= a_0 + a_1 X_{t-1} + \varepsilon_t \\
X_{t-1} &= a_0 + a_1 X_{t-2} + \varepsilon_{t-1} \\
X_{t-2} &= a_0 + a_1 X_{t-3} + \varepsilon_{t-2} \\
&\vdots
\end{aligned}$$

We can recursively substitute X_{t-j} into the equation of X_t, namely,

$$\begin{aligned}
X_t &= a_0 + a_1 X_{t-1} + \varepsilon_t \\
&= a_0 + a_1(a_0 + a_1 X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= a_0(1 + a_1) + a_1^2 X_{t-2} + (\varepsilon_t + a_1 \varepsilon_{t-1}) \\
&= a_0(1 + a_1) + a_1^2(a_0 + a_1 X_{t-3} + \varepsilon_{t-2}) + (\varepsilon_t + a_1 \varepsilon_{t-1}) \\
&= a_0(1 + a_1 + a_1^2) + a_1^3 X_{t-3} + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2}) \\
&= \dots \\
&= a_0(1 + a_1 + a_1^2 + \cdots + a_1^{J-1}) + a_1^J X_{t-J} + (\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + \cdots + a_1^{J-1} \varepsilon_{t-J+1})
\end{aligned}$$

Assuming that $|a_1| < 1$, as $J \to \infty$, the first term becomes

$$\lim_{J \to \infty} a_0(1 + a_1 + a_1^2 + \cdots + a_1^{J-1}) = a_0 \sum_{j=0}^{\infty} a_1^j = \frac{a_0}{1 - a_1}$$

The second term is

$$\lim_{J \to \infty} a_1^J X_{t-J} = 0$$

since $a_1^J \to 0$. Therefore, X_t can be written as a linear process

$$X_t = \frac{a_0}{1 - a_1} + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Mean, variance and autovariance of an AR(1) process (Method I)

Since ε_t is assumed to be iid with mean 0, the mean of X_t can be obtained easily as

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[\frac{a_0}{1 - a_1} + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}\right] = \frac{a_0}{1 - a_1}.$$

The variance of X_t is

$$\text{var}(X_t) = \text{var}\left(\sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}\right) = \sum_{j=0}^{\infty} \text{var}(a_1^j \varepsilon_{t-j}) = \sum_{j=0}^{\infty} a_1^{2j} \sigma^2 = \frac{\sigma^2}{1-a_1^2}.$$

Finally the autocovariance of X_t is given by

$$\begin{aligned}
\gamma_X(h) &= \text{cov}(X_t, X_{t-h}) \\
&= \text{cov}\left(\sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}, \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-h-j}\right) \\
&= \text{cov}\left(\varepsilon_t + a_1\varepsilon_{t-1} + \cdots + a_1^{h-1}\varepsilon_{t-h+1} + \sum_{j=h}^{\infty} a_1^j \varepsilon_{t-j}, \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-h-j}\right) \\
&= \text{cov}\left(\sum_{j=h}^{\infty} a_1^j \varepsilon_{t-j}, \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-h-j}\right) \\
&= \text{cov}\left(a_1^h \varepsilon_{t-h} + a_1^{h+1}\varepsilon_{t-h-1} + \ldots, \varepsilon_{t-h} + a_1\varepsilon_{t-h-1} + \ldots\right) \\
&= (a_1^h \cdot 1 + a_1^{h+1} \cdot a_1 + a_1^{h+2} \cdot a_1^2 + \ldots)\sigma^2 \\
&= \sigma^2 \sum_{j=0}^{\infty} a_1^{2j+h} \\
&= \frac{a_1^h \sigma^2}{1-a_1^2}.
\end{aligned}$$

Therefore, we can also obtain

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = a_1^h.$$

Mean, variance and autovariance of an AR(1) process (Method II)

From the above, we observe that when $|a_1| < 1$, then X_t is weakly stationary. We can then find the mean, variance and autocovariance of X_t more easily using the properties of weak stationarity.

First, we know that $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu_X$. Therefore, taking expectation of Eq.(1),

$$\begin{aligned}
\mu_X = \mathbb{E}[X_t] &= a_0 + a_1\mathbb{E}[X_{t-1}] + \mathbb{E}[\varepsilon_t] \\
&= a_0 + a_1\mu_X \\
&= \frac{a_0}{1-a_1}.
\end{aligned}$$

Now if we substitute $a_0 = \mu_X(1-a_1)$ into Eq.(1) (or alternatively subtract the mean from Eq.(1)), then we have

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \varepsilon_t. \tag{2}$$

Taking square and then expectation of the above equation, we have

$$
\begin{aligned}
\mathbb{E}\left[(X_t-\mu_X)^2\right] = \operatorname{var}(X_t) &= \mathbb{E}\left[(a_1(X_{t-1}-\mu_X)+\varepsilon_t)^2\right] \\
&= a_1^2\mathbb{E}\left[(X_{t-1}-\mu_X)^2\right] + 2\mathbb{E}\left[(X_{t-1}-\mu_X)\varepsilon_t\right] + \mathbb{E}\left[\varepsilon_t^2\right] \\
&= a_1^2 \operatorname{var}(X_t) + \sigma^2 \\
&= \frac{\sigma^2}{1-a_1^2}.
\end{aligned}
$$

Multiplying $X_{t-h}-\mu_X$ to Eq.(2) and taking expectation,

$$
\begin{aligned}
\mathbb{E}\left[(X_t-\mu_X)(X_{t-h}-\mu_X)\right] &= \gamma_X(h) \\
&= a_1\mathbb{E}\left[(X_{t-1}-\mu_X)(X_{t-h}-\mu_X)\right] + \mathbb{E}\left[\varepsilon_t(X_{t-h}-\mu_X)\right] \\
&= a_1\gamma_X(h-1) + \mathbb{E}\left[\varepsilon_t(X_{t-h}-\mu_X)\right]
\end{aligned}
$$

Note that $\mathbb{E}\left[\varepsilon_t(X_{t-h}-\mu_X)\right] = \sigma^2$ when $h=0$ and $\mathbb{E}\left[\varepsilon_t(X_{t-h}-\mu_X)\right] = 0$ when $h>0$,

$$
\gamma_X(h) = a_1\gamma_X(h-1) + \begin{cases} \sigma^2 & \text{if } h=0 \\ 0 & \text{if } h>0 \end{cases}
$$

Using the fact that $\gamma_X(h)=\gamma_X(-h)$, for $h=0$ and $h=1$,

$$
\begin{aligned}
\gamma_X(0) &= a_1\gamma_X(1) + \sigma^2 \\
\gamma_X(1) &= a_1\gamma_X(0)
\end{aligned}
$$

Substituting the second equation to the first, we obtain

$$
\gamma_X(0) = \frac{\sigma^2}{1-a_1^2}, \qquad \gamma_X(h) = a_1\gamma_X(h-1) = a_1^h\gamma_X(0).
$$

Similarly, we can obtain the autocorelation as

$$
\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = a_1^h.
$$