Capital University of Economics and Business

ISEM

Financial Econometrics

AR(p) Process

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AR(p) process

We say X_t is an AR(p) process if it takes the form

$$X_t = a_0 + a_1 X_{t-1} + + \cdots + a_p X_{t-p} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right). \tag{1}$$

Mean, variance and autovariance of a weakly stationary AR(p) process

If X_t is weakly stationary, we can find the mean, autocovariances of X_t easily using the properties of weak stationarity.

First, we know that $\mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_{t-1}\right] = \mu_X$. Therefore, taking expectation of Eq.(1),

$$\begin{aligned} \mu_X = \mathbb{E}\left[X_t\right] &= a_0 + a_1 \mathbb{E}\left[X_{t-1}\right] + \cdots + a_p \mathbb{E}\left[X_{t-p}\right] + \mathbb{E}\left[\varepsilon_t\right] \\ &= a_0 + a_1 \mu_X + \cdots + a_p \mu_X \\ &= \frac{a_0}{1 - a_1 - \cdots - a_p}. \end{aligned}$$

Now if we substitute $a_0 = \mu_X(1 - a_1 - \cdots - a_p)$ into Eq.(1) (or alternatively subtract the mean from Eq.(1)), then we have

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \cdots + a_p(X_{t-p} - \mu_X) + \varepsilon_t. \tag{2}$$

Multiplying $X_{t-h} - \mu_X$ to Eq.(2) and taking expectation,

$$\begin{aligned} \mathbb{E}\left[(X_t - \mu_X)(X_{t-h} - \mu_X)\right] &= \gamma_X(h) \\ &= a_1 \mathbb{E}\left[(X_{t-1} - \mu_X)(X_{t-h} - \mu_X)\right] + \ldots \\ &\qquad + a_p \mathbb{E}\left[(X_{t-p} - \mu_X)(X_{t-h} - \mu_X)\right] + \mathbb{E}\left[\varepsilon_t (X_{t-h} - \mu_X)\right] \\ &= a_1 \gamma_X(h-1) + \cdots + a_p \gamma_X(h-p) + \mathbb{E}\left[\varepsilon_t (X_{t-h} - \mu_X)\right] \end{aligned}$$

Note that $\mathbb{E}\left[\varepsilon_t (X_{t-h} - \mu_X)\right] = \sigma^2$ when $h = 0$ and $\mathbb{E}\left[\varepsilon_t (X_{t-h} - \mu_X)\right] = 0$ when $h > 0$,

$$\gamma_X(h) = a_1 \gamma_X(h-1) + \cdots + a_p \gamma_X(h-p) + \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h > 0 \end{cases}$$

These equations are called the *Yule-Walker equations*. If we now divide $\gamma_X(0)$ from both sides of the equations, then

$$\begin{aligned} \sigma^2 &= \gamma_X(0)(1 - a_1 \rho_X(1) - \cdots - a_p \rho_X(p)) \\ \rho_X(h) &= a_1 \rho_X(h-1) + \cdots + a_p \rho_X(h-p), \qquad h > 0. \end{aligned}$$

These equations are useful in the following two ways:

1. If we know $\gamma_X(0), \ldots, \gamma_X(p-1)$, then we can obtain $\gamma_X(p)$ from the Yuel-Walker equation. Similarly, we can also obtain $\gamma_X(h)$ for every h without needing to estimate them.

2. In reality, $\{a_0, \ldots, a_p\}$ and σ^2 are not known and need to be estimated. From the above equations, we can estimate the parameters from the sample autocorrelations or sample autocovariances.

Yule-Walker estimators

Now consider a weakly stationary AR(2) process

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t, \qquad \varepsilon \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Using the Yule-Walker equations (for autocorrelation) with $h = 1, 2$, we have

$$\begin{aligned} \rho_X(1) &= a_1 \rho_X(0) + a_2 \rho_X(-1) = a_1 + a_2 \rho_X(1) \\ \rho_X(2) &= a_1 \rho_X(1) + a_2 \rho_X(0) = a_1 \rho_X(1) + a_2 \end{aligned}$$

where we use the facts that $\rho_X(0) = 1$ and $\rho_X(-h) = \rho_X(h)$. Writing in matrix form,

$$\begin{pmatrix} \rho_X(1) \\ \rho_X(2) \end{pmatrix} = \begin{pmatrix} 1 & \rho_X(1) \\ \rho_X(1) & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Solving for a_1 and a_2 yields

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 1 & \rho_X(1) \\ \rho_X(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_X(1) \\ \rho_X(2) \end{pmatrix} \\ &= \frac{1}{1 - \rho_X(1)^2} \begin{pmatrix} 1 & -\rho_X(1) \\ -\rho_X(1) & 1 \end{pmatrix} \begin{pmatrix} \rho_X(1) \\ \rho_X(2) \end{pmatrix} \\ &= \frac{1}{1 - \rho_X(1)^2} \begin{pmatrix} \rho_X(1) - \rho_X(1)\rho_X(2) \\ -\rho_X(1)^2 + \rho_X(2) \end{pmatrix} \end{aligned}$$

Therefore,

$$a_1 = \frac{\rho_X(1)(1 - \rho_X(2))}{1 - \rho_X(1)^2}, \qquad a_2 = \frac{\rho_X(2) - \rho_X(1)^2}{1 - \rho_X(1)^2}.$$

The coefficients a_1 and a_2 can therefore be estimated using the sample autocorrelations.

Stationarity of an AR(p) process

Using the lag operator L, we can express an AR(1) process as

$$(1 - a_1 L)(X_t - \mu_X) = \varepsilon_t.$$

It can be interpreted as a stochastic process that *becomes a white noise after applying the filter* $(1 - a_1 L)$. We know that X_t is weakly stationary if $|a_1| < 1$.

For an AR(p) process, we can write it as

$$(1 - a_1 L - \cdots - a_p L^p)(X_t - \mu_X) = \varepsilon_t.$$

If the solution to the root-finding problem

$$1 - a_1 L - \cdots - a_p L^p = 0$$

are given by $r_1^{-1}, \dots, r_p^{-1}$, then we can decompose the polynomial as

$$1 - a_1 L - \cdots - a_p L^p = (1 - r_1 L) \dots (1 - r_p L).$$

Therefore, X_t is given by

$$(1 - r_1 L) \dots (1 - r_p L)(X_t - \mu_X) = \varepsilon_t,$$

It can be interpreted as a process that *becomes a white noise after applying p filters* $(1 - r_j L)$, $j = 1, \dots, p$. It can be shown that X_t is weakly stationary if the solutions to the above root-finding problem are all larger than one in absolute value, i.e., $|r_j^{-1}| > 1$, or $|r_j| < 1$ for all j.

Example

Consider the AR(2) process

$$X_t = 0.4 X_{t-1} - 0.04 X_{t-2} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

We can write the model as

$$(1 - 0.4L + 0.04L^2) X_t = \varepsilon_t.$$

Therefore, to find whether X_t is stationary, we need to find the roots of the lag polynomial. Solving

$$1 - 0.4L + 0.04L^2 = 0$$

yields

$$L = \frac{0.4 \pm \sqrt{0.4^2 - 4 \times 0.04}}{-0.08} = 5 > 1.$$

Therefore, the process is weakly stationary. Besides, the process can be written as

$$(1 - 0.2L)^2 X_t = \varepsilon_t.$$

That is, the process becomes a white noise after apply the filter $1 - 0.2L$ twice.