CAPITAL UNIVERSITY OF ECONOMICS AND BUSINESS

ISEM

Financial Econometrics

Assignment 1 (Solution)

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1 True/false questions

State whether each of the following statements is true or false.

Q1. The average return is the mean of past one-period returns.

> **False**. The average return is the *geometric average* of past one-period returns, not the *simple average*. The average *log return* is the mean of past one-period *log returns*.

Q2. If the log return of an asset is constant, then its price will grow linearly.

> **False**. If the log return is constant, say equals r, then the price at time T will be
>
> $$P_T = P_0 \prod_{t=1}^{T}(1+R_t) = P_0 \prod_{t=1}^{T} \mathrm{e}^r = P_0 \mathrm{e}^{rT}.$$
>
> Therefore, the price will be growing *exponentially*.

Q3. Let X be a random variable. The kurtosis of $10X$ is larger than that of X since the dispersion of $10X$ is larger than that of X.

> **False**. The kurtosis of a random variable is scale-free, i.e., the kurtosis does not change by scaling a variable, since it is the fourth-moment of the *standardized* variable.

Q4. The log return must not normally distributed, since its lower bound is -1.

> **False**. Let r be the log return, then the simple return is given by $1 + R = \mathrm{e}^r$. Although the support of r does not have a lower bound,
>
> $$\lim_{r \to -\infty}(1+R) = \lim_{r \to -\infty} \mathrm{e}^r = 0.$$
>
> Therefore, the lower bound of R is -1.

Q5. It is possible that the loss of an asset is larger than its value-at-risk.

> **True**. The α-VaR, VaR_α, is the maximum loss of an asset with a probability $\alpha\%$, i.e., there is a $1 - \alpha\%$ probability that the value of an asset is larger than VaR_α. There is still α probability that the resulting asset value is below VaR_α.

Q6. The expected shortfall of an asset must be smaller than or equal to the respective value-at-risk.

> **True**. The expected shortfall is the expected value of an asset, *conditional on that the value of the asset is smaller than the value-at-risk*.

Q7. If a stochastic process is strictly stationary, then it is also weakly stationary.

True. If a stochastic process $\{X_t\}$ is strictly stationary, then its density function is time-invariant. Therefore, its mean and variance are also time-invariant. Besides, the joint density between X_t and X_{t-h} is also time-invariant for any h. Therefore, the autocovariance of the stochastic process is also time-invariant. Thus, the process is weakly stationary.

Q8. A linear process may not be weakly stationary.

False. The mean, variance and autocovariance of a linear process are time-invariant. Moreover, its variance is finite by definition. Thus, it must be weakly stationary.

2 Multiple choice questions

Q1. Which of the following gives the highest effective interest rate?

(A) 10% per annum interest rate, compounded annually

(B) 11% per annum interest rate, compounded semi-annually

(C) 8.5% per annum interest rate, compounded monthly

(D) 8% per annum interest rate, compounded daily

B. The effective interest rates are respectively

$$(1+10\%)^1 - 1 = 10\%$$
$$\left(1+\frac{11\%}{2}\right)^2 = 11.3025\%$$
$$\left(1+\frac{8.5\%}{12}\right)^{12} = 8.8391\%$$
$$\left(1+\frac{8\%}{365}\right)^{365} = 8.3275\%$$

Q2. Suppose X is random variable with mean $\mathbb{E}[X] = 1$ and variance $\text{var}(X) = 1$. Find the second moment $\mathbb{E}[X^2]$.

(A) 1

(B) 0

(C) 4

(D) 2

D. Notice that

$$\text{var}(X) = \mathbb{E}\left[(X-\mu)^2\right] = \mathbb{E}\left[X^2\right] - 2\mathbb{E}[X]\mu + \mu^2 = \mathbb{E}\left[X^2\right] - \mu^2$$

Therefore, $\mathbb{E}[X^2] = \text{var}(X) + \mu^2 = 2$.

Q3. If X_t is a weakly stationary stochastic process, then which of the following must be time-invariant?

(A) The joint density function $f_{X_t, X_{t-1}}(x_1, x_2)$.

(B) The kurtosis $K(X_t)$

(C) The expectation of change $\mathbb{E}[X_t - X_{t-1}]$

(D) None of the above

> **C**. The joint density and kurtosis are time-invariant if X_t is strictly stationary. They *can* be time-varying even if X_t is weakly stationary. The expectation of change is $\mathbb{E}[X_t - X_{t-1}] = \mathbb{E}[X_t] - \mathbb{E}[X_{t-1}] = 0$, so it must be time-invariant.

Q4. If$X_t \overset{\text{iid}}{\sim} (\mu, \sigma^2)$, then the following is time-invariant:

(A) Variance of X_t

(B) Kurtosis of X_t

(C) Skewness of X_t

(D) All of the above

> **D**. If X_t is iid, then its density function is time-invariant. Therefore, all moments are time-invariant if they exist.

3 Short questions

Q1. U-quadratic distribution

Let X be a random variable with density function

$$f(x|b, \alpha, \beta) = \alpha(x - \beta)^2, \qquad x \in [0, b]$$

where $b > 0$ and

$$\alpha = \frac{12}{b^3}, \qquad \beta = \frac{b}{2}.$$

Find the mean and variance of X.

The mean of X is given by

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^b \alpha(x - \beta)^2 x dx \\
&= \alpha \int_0^b (x^3 - 2\beta x^2 + \beta^2 x) dx \\
&= \alpha \left[\frac{x^4}{4} - \frac{2\beta}{3} x^3 + \frac{\beta^2}{2} x^2 \right]_0^b \\
&= \frac{12}{b^3} \left[\frac{b^4}{4} - \frac{2\beta b^3}{3} + \frac{\beta^2 b^2}{2} \right] \\
&= 3b - 4b + 2b \\
&= b.
\end{aligned}$$

The variance of X is given by

$$\begin{aligned}
\operatorname{var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= \int_0^b \alpha(x-\beta)^4 dx \\
&= \left[\frac{\alpha}{5}(x-\beta)^5\right]_0^b \\
&= \frac{12}{5b^3}\left[\left(b - \frac{b}{2}\right)^5 - \left(0 - \frac{b}{2}\right)^5\right] \\
&= \frac{3}{20}b^2.
\end{aligned}$$

Q2. Linear process

Consider the linear process

$$X_t = \mu + \sum_{j=0}^{\infty} b_j e_{t-j}, \qquad e_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Suppose that

$$b_j = c^j + d^j$$

where $|c| < 1$ and $|d| < 1$. Find the mean, variance and autocovariance of X_t. Is X_t weakly stationary?

Since e_t is iid, $\mathbb{E}[X_t] = \mu$. The variance can be obtained as

$$\begin{aligned}
\operatorname{var}(X_t) &= \sum_{j=0}^{\infty} b_j^2 \sigma^2 \\
&= \sigma^2 \sum_{j=0}^{\infty} (c^j + d^j)^2 \\
&= \sigma^2 \sum_{j=0}^{\infty} (c^{2j} + 2(cd)^j + d^{2j}) \\
&= \sigma^2 \left(\frac{1}{1-c^2} + \frac{2}{1-cd} + \frac{1}{1-d^2}\right).
\end{aligned}$$

The autocovariance function is given by

$$\begin{aligned}
\gamma_X(h) &= \sigma^2 \sum_{j=0}^{\infty} b_j b_{j+h} \\
&= \sigma^2 \sum_{j=0}^{\infty} (c^j + d^j)(c^{j+h} + d^{j+h}) \\
&= \sigma^2 \sum_{j=0}^{\infty} (c^{2j+h} + c^j d^{j+h} + d^j c^{j+h} + d^{2j+h}) \\
&= \sigma^2 \left(\frac{c^h}{1-c^2} + \frac{c^h + d^h}{1-cd} + \frac{d^h}{1-d^2}\right)
\end{aligned}$$

Since the mean, variance and autocovariance are time-invariant, the stochastic process X_t is weakly stationary.