Capital University of Economics and Business

ISEM

Financial Econometrics

Assignment 2 (Solution)

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1 True/false questions

State whether each of the following statements is true or false.

Q1. An AR(1) process is always weakly stationary.

> **False**. An AR(1) process is weakly stationary only if the AR coefficient is smaller than one in absolute value.

Q2. We can always estimate the coefficients in an AR process using the Yule-Walker equations.

> **False**. We can only use the Yule-Walker equations to estimate the coefficients only if the AR process is weakly stationary.

Q3. The autocorrelation function of an AR(p) process cuts off at lag p.

> **False**. The *partial* autocorrelation function of an AR(p) process cuts off at lag p.

Q4. The AR order selected by BIC tends to be smaller than that selected by AIC.

> **True**. Since the penalty term of BIC is larger than that of AIC as long as $\ln T > 2$, or $T > 7$, BIC tends to choose a smaller lag order.

Q5. The variance of h-step-ahead forecast error increases with h in an AR process.

> **True**. The current information set at time t, $\mathcal{I}_t$, carries less and less information as h increases. Therefore, the forecast error increases with h.

Q6. A model is adequate if the fitted residuals do not have serial correlation.

> **True**. This is the definition of an adequate model.

Q7. Any MA process can be written as an AR(∞) process.

> **False**. Only invertible MA processes can be written as an AR(∞) process.

Q8. An ARMA model is weakly stationary only if the MA part is invertible.

> **False**. An ARMA model is weakly stationary only if the model is casual.

2 Multiple choice questions

Q1. Consider an AR(1) process $X_t = a_0 + a_1 X_{t-1} + \varepsilon_t$, where $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$ and $|a_1| < 1$. Which of the following is true?

(A) X_t is weakly stationary.

(B) $\gamma_X(h) \to 0$ as $h \to \infty$.

(C) $\mathbb{E}[X_t] = 0$ if and only if $a_0 = 0$.

(D) All of the above.

> **D**. An AR(1) process is weakly stationary if and only if $|a_1| < 1$. In this case, the autocovaraince function is $\gamma_X(h) = a_1^h \gamma_X(0) \to 0$ as $h \to \infty$. Finally, the unconditional mean of X_t is $\mu_X = a_0(1 - a_1)^{-1}$, which equals zero if and only if $a_0 = 0$.

Q2. Suppose you are trying to select the order of an AR(p) process. What is the suggested order according to the following table?

j	0	1	2	3	4
AIC(j)	0.066	-0.012	-0.039	-0.037	-0.034

(A) 0

(B) 1

(C) 2

(D) 3

> **C**. We select the AR order by minimizing the AIC. Therefore, we choose $p = 2$.

Q3. Consider an AR(1) process $X_t = a_0 + a_1 X_{t-1} + \varepsilon_t$, where $\varepsilon_t \overset{\text{iid}}{\sim} (0, 1)$. Suppose also that the variance of X_t is $\gamma_X(0) = 2$. Find a_1.

(A) $\sqrt{0.5}$

(B) $-\sqrt{0.5}$

(C) All of the above.

(D) None of the above.

> **C**. Since $\gamma_X(0) = \sigma^2/(1 - a_1^2)$, solving for a_1 gives $a_1 = \pm\sqrt{1 - \sigma^2/\gamma_X(0)} = \pm\sqrt{0.5}$. Therefore, a_1 can be either $\sqrt{0.5}$ or $-\sqrt{0.5}$.

Q4. Consider an ARMA(1,1) process $X_t = a_1 X_{t-1} + \varepsilon_t - b_1 \varepsilon_{t-1}$, $\varepsilon \overset{\text{iid}}{\sim} (0, \sigma^2)$. Which of the following statements are equivalent?

(1) X_t is weakly stationary.

(2) $|a_1| < 1$.

(3) $|b_1| < 1$.

(4) $a_1 = b_1$.

(A) (1) and (2) only.

(B) (1) and (3) only.

(C) (1) and (4) only.

(D) (2) and (3) only.

A. An ARMA(1,1) process is weakly stationary if and only if $|a_1| < 1$. Note that the stationarity condition of an ARMA process is the same as that of an AR process. If $|b_1| < 1$, then the process is invertible, but it does not affect the stationarity property of the process. If $a_1 = b_1$, then the process becomes a white noise sequence.

3 Short questions

Q1. Forecasting an AR(1) process

Consider the AR(1) process

$$X_t = a_0 + a_1 X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{U}(-1,1).$$

(a) Find the variance of a random variable with uniform distribution $\varepsilon \sim \mathcal{U}(-1,1)$.

(b) Find the h-step-ahead forecast of X_t for $h = 1, 2$.

(c) What is the variance of the h-step-ahead forecast error?

Hint: The density function of a uniformly distributed function $\varepsilon \sim \mathcal{U}(a,b)$ is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

(a) First, the mean of ε is given by

$$\mathbb{E}[\varepsilon] = \int_{-1}^{1} \varepsilon f_\varepsilon(\varepsilon) d\varepsilon = \int_{-1}^{1} \varepsilon \frac{1}{2} d\varepsilon = \left[\frac{1}{4}\varepsilon^2\right]_{-1}^{1} = 0.$$

Therefore, the variance is

$$\text{var}(\varepsilon) = \mathbb{E}\left[\varepsilon^2\right] = \int_{-1}^{1} \varepsilon^2 \frac{1}{2} d\varepsilon = \left[\frac{1}{6}\varepsilon^3\right]_{-1}^{1} = \frac{1}{3}.$$

(b) The one-step ahead forecast is

$$\hat{X}_t(1) = \mathbb{E}[X_{t+1}|X_t] = a_0 + a_1 X_t.$$

The two-step ahead forecast is

$$\hat{X}_t(2) = \mathbb{E}[X_{t+2}|X_t] = a_0 + a_1 \hat{X}_t(1) = a_0(1 + a_1) + a_1^2 X_t.$$

(c) The one-step ahead forecast error is

$$\hat{e}_t(1) = X_{t+1} - \hat{X}_t(1) = \varepsilon_{t+1}.$$

Therefore, the variance is $\text{var}(\hat{e}_t(1)) = 1/3$. The two-step ahead forecast error is

$$\begin{aligned}\hat{e}_t(2) &= X_{t+2} - \hat{X}_t(2) \\ &= (a_0 + a_1 X_{t+1} + \varepsilon_{t+2}) - (a_0 + a_1 \hat{X}_t(1)) \\ &= a_1 \hat{e}_t(1) + \varepsilon_{t+2} \\ &= a_1 \varepsilon_{t+1} + \varepsilon_{t+2}.\end{aligned}$$

Therefore the variance is $\text{var}(\hat{e}_t(2)) = (a_1^2 + 1)/3$.

Q2. MA representation of an ARMA process

Consider the ARMA(1,1) process

$$X_t = a_1 X_{t-1} + \varepsilon_t - b_1 \varepsilon_{t-1}, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Express X_t as an MA(∞) process.

Express the process in lag polynomial,

$$(1 - a_1 L) X_t = (1 - b_1 L) \varepsilon_t.$$

Applying the lag polynomial $(1 - a_1 L)^{-1}$ to both sides,

$$\begin{aligned}X_t &= (1 - a_1 L)^{-1} (1 - b_1 L) \varepsilon_t \\ &= (1 + a_1 L + a_1^2 L^2 + \dots)(1 - b_1 L) \varepsilon_t \\ &= (1 + (a_1 - b_1) L + (a_1^2 - a_1 b_1) L^2 + \dots) \varepsilon_t \\ &= \left[1 + \sum_{j=1}^{\infty} a_1^{j-1} (a_1 - b_1) L^j\right] \varepsilon_t.\end{aligned}$$