Capital University of Economics and Business

ISEM

Financial Econometrics

Stationarity of an MA(1) Process

Instructor:
CHEUNG Ying Lun

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Consider the stochastic process

$$X_t = e_t + e_{t-1}, \qquad e_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1).$$

Find the expected value and autocovariance of X_t. Is X_t weakly stationary? Is X_t strictly stationary?

The expected value of X_t is given by

$$\mathbb{E}[X_t] = \mathbb{E}[e_t + e_{t-1}] = \mathbb{E}[e_t] + \mathbb{E}[e_{t-1}] = 0.$$

The variance of X_t is given by

$$\text{var}(X_t) = \mathbb{E}\left[(X_t - \mathbb{E}[X_t])^2\right] = \mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[(e_t + e_{t-1})^2\right]$$

Since e_t is an iid (*independently and identically distributed*) process, $\mathbb{E}[e_t e_{t-1}] = 0$. Therefore, expanding the bracket we have

$$\text{var}(X_t) = \mathbb{E}\left[e_t^2 + 2e_t e_{t-1} + e_{t-1}^2\right] = 2.$$

Similarly, the first-order autocovariance of X_t is

$$\gamma_X(1) = \text{cov}(X_t, X_{t-1}) = \mathbb{E}[X_t X_{t-1}] = 1.$$

The h-th order autocovariance of X_t for all $h \geq 2$ is

$$\gamma_X(h) = \text{cov}(X_t, X_{t-1}) = 0.$$

Since both the mean and autocovariances of X_t are finite and time-invariant, X_t is *weakly stationary*.

Moreover, since e_t is iid normal, X_t is also normally distributed. For example, the joint distribution of (X_t, X_{t-1}) is given by

$$\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} \overset{\text{iid}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)$$

Here the covariance matrix of (X_t, X_{t-1}) is obtained by

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_t) & \text{cov}(X_t, X_{t-1}) \\ \text{cov}(X_{t-1}, X_t) & \text{var}(X_{t-1}) \end{pmatrix} = \begin{pmatrix} \sigma_X^2 & \gamma_X(1) \\ \gamma_X(1) & \sigma_X^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since the joint distribution of (X_t, X_{t-1}) (and actually any combinations of $\{X_t\}$) is identical for any t, X_t is also *strictly stationary*.