Applied Stochastic Process

Ia Probability and Random Variable

CHEUNG Ying Lun

Capital University of Economics and Business

Definition

A random variable is a measurable function $X : \Omega \to \mathbb{R}$ from Ω to the real line, where Ω is a sample space of the probability triple $(\Omega, \mathcal{F}, P)$.

Probability Space

Probability Space

Definition

A probability space, or a probability triple $(\Omega, \mathcal{F}, P)$, consists of three elements:

- The **sample space** Ω which is a nonempty set that contains all possible outcomes.
- The **event space** $\mathcal{F}$ which is a collection of subsets of Ω that represents the events we want to consider.
- The **probability function** $P : \mathcal{F} \to [0, 1]$ which assigns probabilities to each event in $\mathcal{F}$.

Example

Suppose we are tossing a fair coin, denote a head as H and a tail as T, then we can define the respective probability space:

- The sample space is $\Omega = \{H, T\}$.
- The event space is $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$.
- The probability function
  - $P(\emptyset) = 0$
  - $P(\{H\}) = P(\{T\}) = 0.5$
  - $P(\{H, T\}) = 1$

Example

Suppose we are rolling a fair die, then we can define the respective probability space:

- The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- The event space is $\mathcal{F} = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$.
- The probability function
  - $P(\emptyset) = 0$
  - $P(\{1, 3, 5\}) = P(\{2, 4, 6\}) = 0.5$
  - $P(\Omega) = 1$

Definition

Given the sample space Ω, $\mathcal{F}$ is a σ-algebra of Ω if it is a set of subsets of Ω that satisfies the following conditions:

- $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then its complement A^C is also in $\mathcal{F}$.
- If $\{A_1, A_2, \ldots\}$ is a countable collection of sets in $\mathcal{F}$, then so is their union $\cup_i A_i \in \mathcal{F}$.

Assumption

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, we make the following assumptions:

1. The probability of any event A is a non-negative real number, i.e., $P(A) \geq 0$.
2. The probability of at least one of all the possible outcomes of a process will occur is one, i.e., $P(\Omega) = 1$.
3. If two events A and B are *mutually exclusive*, then $P(A \cup B) = P(A) + P(B)$.

Union, intersection and complement

- Union: $A \cup B$ is true when at least one of A *or* B happens.
- Intersection: $A \cap B$ is true when *both* A *and* B happen at the same time.
- Complment: A^C is true when A does not happen, i.e., $A \cup A^C = \Omega$ and $A \cap A^C = \emptyset$.

Exercise

Suppose we are rolling a fair die. What is the probability that:

1. The die is either odd or larger than 4?
2. The die is both odd and larger than 4?
3. The die is not odd?

Let $A|B$ denote the event that A happens, conditional on B happening. The probability of $A|B$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

What is the probability that a die is 6, given that the die is even?

If A and B are independent, then the condition that B happens contains not information on whether A happens or not.
Therefore,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$
$$\implies P(A \cap B) = P(A)P(B).$$

Random Variable and Distributions

Random variable

Definition

A random variable is a measurable function $X : \Omega \to \mathbb{R}$ from Ω to the real line, where Ω is a sample space of the probability triple $(\Omega, \mathcal{F}, P)$.

Bernoulli distribution

Consider tossing a coin, suppose the probability of a head is p, and let $\Omega = \{T, H\}$ be the sample space. If we let

$$X(\{T\}) = 0, \qquad X(\{H\}) = 1.$$

Then,

$$\begin{aligned} P(X = 0) &= P(\omega = T) = 1 - p \\ P(X = 1) &= P(\omega = H) = p \end{aligned} \tag{1}$$

and we say X has the Bernoulli distribution with probability mass function Eq.(1).

Binomial distribution

Suppose we are tossing a coin for n times, and denote X as the number of heads. Then the probability mass function of X is

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \ldots, n$$

where

$$\binom{n}{i} = \frac{n!}{(n-i)!i!}$$

We say X is a binomial random variable with prarameters (n, p), or simply $X \sim Bin(n, p)$.

Suppose $X_n \sim Bin(n, p)$ with $\lambda = np$, or equivalently $p = \lambda/n$. Let $X = \lim_{n\to\infty} X_n$, when $n \to \infty$ and $p \to 0$, we can show that

$$P(X = i) = \mathrm{e}^{-\lambda}\frac{\lambda^i}{i!}, \qquad i = 0, 1, \ldots$$

We say X has the Poisson distribution with parameter λ, or $X \sim Poi(\lambda)$. When is the Poisson distribution useful?

Geometric distribution

Suppose we are tossing a coin until we get a head, and let X be the number of trials. Then, the probability mass function of X is

$$P(X = i) = (1 - p)^{i-1} p, \qquad i = 1, 2, \ldots$$

We say X is a geometric random variable with parameter p, or $X \sim Geo(p)$.

Suppose $X_n \sim Geo(p_n)$ where $p_n = \lambda/n$ and $X = \lim_{n\to\infty} n^{-1} X_n$, then one can show that

$$F_X(t) = P(X \leq t) = 1 - \mathrm{e}^{-\lambda t}, \qquad t \geq 0.$$

We say X is exponentially distributed with parameter λ, or$X \sim exp(\lambda)$. Note that X is a *continuous* random variable. Its probability density function is given by

$$f_X(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Meaning of each distribution

Distribution	Useful for modeling the...
Bernoulli	success of a single event
Binomial	number of times of success among n independent trials
Poisson	number of times of success in an interval of time
Germetric	number of independent trials until the first success
Exponential	time until the first success