Applied Stochastic Process

Ib Expectation

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Expecatation of a Random Variable

Definition

If X is a discrete random variable with possible values $\{x_1, x_2, \ldots\}$ and probability $P(X = x_i) = p_X(x_i)$, then the expectation of X is

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i p_X(x_i).$$

If X is a continuous random variable with support $(-\infty, \infty)$ and probability density function $f_X(x)$, then its expectation is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Exercise

Let X be the outcome when we roll a fair die, find

1. $\mathbb{E}\left[X\right]$
2. $\mathbb{E}\left[X^2\right]$

Let $Y = g(X)$. Then, if X is discrete,

$$\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_{i=1}^{\infty} g(x_i) p_X(x_i).$$

If X is continuous,

$$\mathbb{E}[Y] = \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Moments of a random variable

- Mean $\mu = \mathbb{E}\left[X\right]$: Measure of the central tendency
- Variance $\sigma^2 = \mathbb{E}\left[(X-\mu)^2\right]$: Measure of dispersion
- Skewness $s = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$: Measure of asymmetry
- Kurtosis $\kappa = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$: Measure of tailedness

Jointly Distributed Random Variables and Conditioning

For any two random variables X and Y, their joint cumulative probability function is defined by

$$F(a,b) = P(X \le a, Y \le b), \qquad -\infty < a, b < \infty.$$

Their joint probability mass function is defined by

$$p(x,y) = P(X = x, Y = y)$$

What is the cumulative probability function and probability mass function of X?

For any two *jointly continuous* random variables X and Y with joint probability density function $f(x,y)$,

$$P(X \in \mathcal{A}, Y \in \mathcal{B}) = \int_{\mathcal{B}} \int_{\mathcal{A}} f(x,y) dx dy.$$

The probability density function of X can be obtained by

$$\begin{aligned} P(X \in \mathcal{A}) &= P(X \in \mathcal{A}, Y \in (-\infty, \infty)) \\ &= \int_{-\infty}^{\infty} \int_{\mathcal{A}} f(x,y) dx dy \\ &= \int_{\mathcal{A}} f_X(x) dx \end{aligned}$$

For any function g of two variables,

$$\mathbb{E}\left[g(X,Y)\right] = \sum_y \sum_x g(x,y)p(x,y) \qquad \text{(discrete case)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy \qquad \text{(continuous case)}$$

Exercise

1. Let $g(X, Y) = aX + bY$, $\mathbb{E}[X] = \mu_X$ and $\mathbb{E}[Y] = \mu_Y$.
   1.1 Find $\mathbb{E}[g(X, Y)]$.
   1.2 Find $\text{var}(g(X, Y))$.
2. Let $X_1, \ldots, X_n$ be n independent Bernoulli random variables with probability of success p. Find the expectation of the sum of all X_i.

Definition

The random variables X and Y are said to be independent if for all a, b

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b).$$

Independent random variables

If X and Y are independent, then

- $F(a, b) = F_X(a)F_Y(b)$
- $p(x, y) = p_X(x)p_Y(y)$ if they are discrete
- $f(x, y) = f_X(x)f_Y(y)$ if they are continuous
- $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$ for any functions h and g

What is the covariance between X and Y in this case?

Exercise

Suppose X_i, $i = 1, \ldots, N$ are mutually independent with the same mean μ and variance σ^2. Find the mean and variance of the sample mean

$$\overline{X} = N^{-1} \sum_{i=1}^{N} X_i.$$

Theorem

Let $X_1, X_2, \ldots, X_N$ be a sequence of indpendent random variables with a common distribution and $\mathbb{E}[X_i] = \mu$ for all i. Then, with probability 1,

$$\overline{X} = N^{-1} \sum_{i=1}^{N} X_i \xrightarrow{p} \mu \qquad \textit{as } N \to \infty.$$

Theorem

Let $X_1, X_2, \ldots, X_N$ be a sequence of indpendent identically distributed random variables with mean μ and variance σ^2 for all i. Then as $N \to \infty$,

$$N^{-1/2} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma} = \frac{\sqrt{N}(\overline{X} - \mu)}{\sigma} \xrightarrow{d} \mathcal{N}(0,1)$$

In other words,

$$P\left(\frac{\sqrt{N}(\overline{X} - \mu)}{\sigma} \leq a\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \mathrm{e}^{-x^2/2} dx.$$

Exercise

Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables with parameter p. Let

$$S_N = \sum_{i=1}^{N} X_i.$$

1. Find the mean μ_N and variance σ_N^2 of S_N.
2. Find the distribution of $\sigma_N^{-1}(S_N - \mu_N)$ when $N \to \infty$.