Applied Stochastic Process

1c Stochastic Process - Definitions

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Definition

Let $\mathcal{T}$ be an ordered set and $(\Omega, \mathcal{F}, P)$ a probability space. A stochastic process is a collection of random variables $X = \{X_t; t \in \mathcal{T}\}$ where, for each fixed $t \in \mathcal{T}$, X_t is a random variable from $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}$.

Definition

The stochastic process $\{X(t), t \in \mathcal{T}\}$ is said to be *ergodic* if any characteristics of the process can be obtained, with probability 1, from a single realization.

Mean ergodicity

Let $\overline{X}_T$ be the temporal mean of the stochastic process $\{X(t), t \in \mathcal{T}\}$, defined by

$$\overline{X}_T = \frac{1}{T}\int_0^T X(t)dt, \qquad \text{or } \overline{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t$$

if $\mathcal{T}$ is continuous and discrete respectively. Then, the stochastic process $\{X(t)\}$ for which $\mathbb{E}\left[X_t\right] = \mu_X \ \forall t$ is said to be *mean ergodic* if

$$P\left(\lim_{T\to\infty} \overline{X}_T = \mu_X\right) = 1.$$

Definition

We say that the stochastic process $\{X(t), t \in \mathcal{T}\}$ is stationary, or *strongly stationary*, if the joint distribution function of any order n is invariant under any change of origin:

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = F(x_1, \ldots, x_n; t_1 + s, \ldots, t_n + s)$$

for all s, n and $t_1, \ldots, t_n$.

Definition

We say that the stochastic process $\{X(t), t \in \mathcal{T}\}$ is *weakly stationary* if $\mathbb{E}[X_t] = \mathbb{E}[X_{t+s}] = \mu_X$ for any s and

$$\gamma_X(t_1, t_2) = \gamma_X(t_2 - t_1) \qquad \forall t_1, t_2 \in \mathcal{T}$$

where $\gamma_X(t_1, t_2) = \text{cov}(X_{t_1}, X_{t_2})$ is the autocovariance function of X_t.

Mean ergodicity

Theorem (Ergodic Theorems)

A weakly stationary process $\{X(t), t \in \mathcal{T}\}$ *with mean* μ *and autocovariance function* $\gamma_X(h)$ *is mean ergodic iff*

$$\lim_{T \to \infty} \frac{1}{T} \sum_{h=0}^{T-1} \gamma_X(h) = \operatorname{cov}(\overline{X}_T, X_T) = 0.$$

Two sufficient conditions are

1. $\gamma_X(0) < \infty$ *and* $\lim_{|h| \to \infty} \gamma_X(h) = 0$*;*
2. $\sum_{-\infty}^{\infty} |\gamma_X(h)| < \infty$*.*

Exercise

Are the following processes mean ergodic? Are they (weakly/strongly) stationary?

1. $X_t = X$, where $X \sim \mathcal{N}(0, 1)$.
2. $X_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$.

Definition

A stochastic process $\{X(t), t \in \mathcal{T}\}$ is said to be Markovian if

$$P(X(t_n) \in A | X(t), t \leq t_{n-1}) = P(X(t_n) \in A | X(t_{n-1}))$$

where $t_{n-1} < t_n$.

	Discrete state	Continuous state
Discrete time	Markov chain	Random walk
Continuous time	Poisson process	Brownian motion