Applied Stochastic Process

2a Markov Chain (I)

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Definition

A Markov chain with state space $\mathcal{X}$ is a sequence $X_0, X_1, \ldots$ of $\mathcal{X}$-valued random variables such that for all states $i, j, k_0, k_1, \ldots$ and all times $t = 0, 1, 2, \ldots,$

$$\begin{aligned} P(X_{t+1} = j | X_t = i, X_{t-1} = k_{t-1}, \ldots) &= P(X_{t+1} = j | X_t = i) \\ &= p_{i,j} \end{aligned}$$

where $p_{i,j}$ depends only on the states i, j and not on the time t or the previous states $k_{t-1}, k_{n-2}, \ldots.$

The matrix $\mathbf{P}$ that collects the transition probabilities $p_{i,j}$ is called the *transition matrix*, defined as

$$\mathbf{P} = (p_{i,j})_{i,j} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,M} \\ \vdots & \ddots & \vdots \\ p_{M,1} & \cdots & p_{M,M} \end{pmatrix}$$

If the initial state is random with probability $\mathbf{q}_0 = (q_{01}, \ldots, q_{0M})$, then the unconditional probability at period 1 is given by $\mathbf{q}_1 = \mathbf{q}_0 \mathbf{P}$.

Let $P_{ij}^{n} = P(X_{n+k} = j | X_k = i)$ be the n-step transition probabilities. Then the Chapman-Kolmogorov equations are

$$P_{ij}^{n+m} = \sum_{k=1}^{M} P_{ik}^{n} P_{kj}^{m} \qquad \text{for all } n, m \geq 0, \text{ all } i, j.$$

Moreover, let $\mathbf{P}^{(n)}$ be the matrix of n-step transition probabilities, then

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}.$$

If the initial state is random with probability $\mathbf{q}_0$, then the unconditional probability at period n is given by $\mathbf{q}_n = \mathbf{q}_0 \mathbf{P}^{n}$.

Example

Consier a gambling game in which on any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. Suppose you quit the game either if your fortune reaches $\$N$ or \$0. Let X_n be the amount you have after n plays.

1. Is X_n Markovian?
2. What is the transition matrix?
3. How often does one lose all one's money?
4. On average, how long does one play before quitting the game?

Example

Suppose there are three types of laundry detergent, and let X_n be the brand chosen on the nth purchase. Customers who try these brands are satisfied and choose the same thing again with probability 0.8, 0.6, and 0.4 respectively. When they change they pick on of the other two brands at random.

1. What is the transition probability?
2. Do the market shares of the three product stabilize?

Theorem

The long-run proportions of an irreducible Markov chain are the unique solution of the equations

$$\pi_j = \sum_i \pi_i P_{ij}, \qquad \sum_j \pi_j = 1$$

or in matrix form $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, *if the chain is positive recurrent.*