Applied Stochastic Process

2b Markov Chain (II) - Classification of States

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Definition

1. State j is said to be *accessible* from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.
2. Two states i and j that are accessible to each other are said to *communicate*, and we write $i \leftrightarrow j$.
3. Two states that communicate are said to be in the same *class*.
4. The Markov chain is said to be *irreducible* if there is only one class, i.e., if all states communicate with each other.

Property

- $i \leftrightarrow i$ for all i.
- If $i \leftrightarrow j$, then $j \leftrightarrow i$.
- If $i \leftrightarrow j$, and $j \leftrightarrow k$, then $i \leftrightarrow k$.
- Any two classes of states are either identical or disjoint.

Example

How many classes of states are there in the examples of Gambler's Ruin and Brand Preference?

Definition

For any state i, let $f_{i,j}$ be the probability that, starting in state i, the process will ever enter state j. State i is said to be *recurrent* if $f_{i,i} = 1$ and *transient* if $f_{i,i} < 1$.

Property

- If state i is recurrent, then
  - starting in state i, the process will reenter state i infinitely often.
  - $f_{i,j} = f_{j,j} = 1$ if $i \leftrightarrow j$. Therefore, recurrence is a class property.
- If state i is transient, then starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $1/(1 - f_{i,i})$.
- It is impossible to go from a recurrent to a transient state.
- In a finite state Markov chain, at least one class of state is recurrent.

Example

Which states are transient in the examples of Gambler's Ruin and Brand Preference?

Definition

If state j is recurrent, let m_j be the expected number of transitions that it takes the Markov chain when starting in state j to return to that state. Then, the recurrent state j is said to be *positive recurrent* if $m_j < \infty$ and *null recurrent* if $m_j = \infty$.

Property

Let π_j be the long run proportion of time in state j.

- If the Markov chain is irreducible and recurrent, then for any initial state $\pi_j = 1/m_j$.
- If i is positive recurrent and $i \leftrightarrow j$, then j is positive recurrent.
- Null recurrence is also a class property.
- An irreducible finite state Markov chain must be positive recurrent.

Theorem

The long-run proportions of an irreducible Markov chain are the unique solution of the equations

$$\pi_j = \sum_i \pi_i P_{ij}, \qquad \sum_j \pi_j = 1$$

or in matrix form $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, *if the chain is positive recurrent. Moreover, if there is no solution of the preceding linear equations, then the Markov chain is either transient or null recurrent and all* $\pi_j = 0$.

Definition

The *period* of a state is the largest number that will divide all the $n \geq 1$ for which $p_{i,i}^{(n)} > 0$. A Markov chain is said to be *aperiodic* if all of its states have period one, and it is *periodic* otherwise.

Theorem

- *A periodic Markov chain does not have limiting probabilities.*
- *The limiting probabilities of an irreducible, aperiodic chain always exist and do not depend on the initial state.*
- *The limiting probabilities, when they exist, will equal the long-run proportions.*
- *An irreducible, positive recurrent, aperiodic Markov chain is said to be ergodic.*

Example

In the examples of Gambler's Ruin and Brand Preference, do the stationary distributions exist?