



Applied Stochastic Process

2c Branching Processes

CHEUNG Ying Lun

Capital University of Economics and Business



Letters | 21 April 2020

Reporting, Epidemic Growth, and Reproduction Numbers for the 2019 Novel Coronavirus (2019-nCoV) Epidemic FREE

Ashleigh R. Tuite, PhD, MPH, David N. Fisman, MD, MPH

[Author, Article and Disclosure Information](#)

<https://doi.org/10.7326/M20-0358>

 PDF  Tools  Share

Background: Virologically confirmed cases of 2019 novel coronavirus (2019-nCoV) in China and other countries have increased sharply (1, 2), leading to concerns regarding its pandemic potential. Viral epidemiology has been characterized sufficiently to permit construction of transmission models that predict the future course of this epidemic (3).

Objective: To provide insight into the changing nature of case findings and epidemic growth.

Methods: We developed a simple disease-transmission model in which the 2019-nCoV epidemic was modeled as a [branching process](#) starting in mid-November 2019, with a serial interval of 7 days (time between cases) and a basic reproduction number (R_0) of 2.3 (new cases from each old case), based on available data and assuming no intervention ([Figure 1](#)). The epidemic start date aligned our modeled case counts to point estimates from international case exportation data (4). The model estimated plausible values of the effective reproduction number (R_e ; reproduction number in the presence of control efforts) after implementation of a quarantine in Wuhan and surrounding areas of China on 24 January 2020 (3) ([Figure 1](#)).



Consider a population of reproducing individuals with the following features:

- ▶ Initial population: $X_0 = 1$
- ▶ Lifetime of each individual: 1 unit of time
- ▶ Number of offsprings: $j \geq 0$ with probability P_j , independent of the numbers produced by other individuals

The branching process is $\{X_0, X_1, \dots\} = \{X_n : n \in \mathbb{N}\}$, given by

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}, \quad P(Z_{n,i} = j) = P_j, \quad j \geq 0$$

where $Z_{n,i}$ is the number of offspring produced by individual i at time n .

Some properties:

- ▶ State 0 is a recurrent state, since $P_{00} = 1$.
- ▶ If $P_0 > 0$, then all other states are transient.
- ▶ If $P_0 > 0$, then the population will either die out or its size will diverge to infinity.
- ▶ Every individual in every generation starts a new, independent branching process.

Let μ and σ^2 be the mean and variance of the number of offspring produced by a single individual,

$$\mu = \sum_{j=0}^{\infty} jP_j, \quad \sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j.$$

Then, the expected value and variance of X_n is

$$\mathbb{E}[X_n] = \mu^n, \quad \text{var}(X_n) = \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right), & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}$$

- ▶ **Law of Iterated Expectation (LIE):**

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$$

- ▶ **Law of Total Variance (LTV):**

$$\text{var}(Y) = \mathbb{E}[\text{var}(Y|X)] + \text{var}(\mathbb{E}[Y|X])$$

Conditioning on X_{n-1} ,

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]] \\ &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{i,n} | X_{n-1}\right]\right] \\ &= \mathbb{E}[X_{n-1}\mu] \\ &= \mu\mathbb{E}[X_{n-1}] \\ &= \dots \\ &= \mu^n\mathbb{E}[X_0] \\ &= \mu^n\end{aligned}$$

Conditioning on X_{n-1} ,

$$\mathbb{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \text{var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

By the Law of Total Variance,

$$\begin{aligned}\text{var}(X_n) &= \mathbb{E}[\text{var}(X_n|X_{n-1})] + \text{var}(\mathbb{E}[X_n|X_{n-1}]) \\ &= \mathbb{E}[X_{n-1}\sigma^2] + \text{var}(X_{n-1}\mu) \\ &= \sigma^2\mu^{n-1} + \mu^2 \text{var}(X_{n-1})\end{aligned}$$

By recursive substitution,

$$\text{var}(X_n) = \begin{cases} \sigma^2\mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right), & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}$$

Suppose $Z_{n,i} \sim \text{Geo}(p)$. Compare the branching process when $p = 0.3, 0.5, 0.7$ by answering the following questions.

1. What is $\mathbb{E}[Z_{n,i}]$?
2. How often does the population extinct? How long does it take?
3. What is the mean and variance of X_n , $n = 1, \dots, 20$?