Applied Stochastic Process

3a Poisson Process

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Poisson and Exponential Distributions

Poisson distribution

Suppose $X_n \sim Bin(n, p)$ with $\lambda = np$, or equivalently $p = \lambda/n$. Let $X = \lim_{n\to\infty} X_n$, when $n \to \infty$ and $p \to 0$, we can show that

$$P(X = i) = \mathrm{e}^{-\lambda}\frac{\lambda^i}{i!}, \qquad i = 0, 1, \ldots$$

We say X has the Poisson distribution with parameter λ, or $X \sim Poi(\lambda)$.

Properties

Some properties of the Poisson distribution:

- Moments: $\mathbb{E}[X] = \lambda$, $\mathrm{var}(X) = \lambda$.
- Independent sums: If X_i are independent $Poi(\lambda_i)$, then $X_1 + \cdots + X_n \sim Poi(\lambda_1 + \cdots + \lambda_n)$

Exponential distribution

Suppose $T_n \sim Geo(p_n)$ where $p_n = \lambda/n$ and $T = \lim_{n\to\infty} n^{-1}T_n$, then one can show that

$$F_T(t) = P(T \leq t) = 1 - \mathrm{e}^{-\lambda t}, \qquad t \geq 0.$$

We say T is exponentially distributed with parameter λ, or $T \sim exp(\lambda)$. Its probability density function is given by

$$f_T(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Some properties of the exponential distribution:

- Moments: $\mathbb{E}[T] = 1/\lambda$, $\text{var}(T) = 1/\lambda^2$.
- Lack of memory: $P(T > t + s | T > t) = P(T > s)$.
- Exponential races: If $S \sim exp(\lambda)$ and $T \sim exp(\mu)$ are independent, then $\min\{S, T\} \sim exp(\lambda + \mu)$.

Counting Process

Definition

A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of 'events' that occur by time t.

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Example

- Number of persons who enter a particular store by time t
- Number of people who were born by time t
- Number of goals a football player scores by time t

Property

- $N(t) \geq 0$.
- $N(t)$ is integer valued.
- If $s < t$, then $N(s) \leq N(t)$, i.e., $N(t)$ is increasing monotonically.
- For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

Poisson Process

Definition

$\{N(t), t \geq 0\}$ is a Poisson process if

1. $N(0) = 0$
2. $N(t)$ has *stationary* increments, specifically, $N(t+s) - N(s) \sim Poi(\lambda t)$
3. $N(t)$ has *independent* increments, i.e., for $t_0 < t_1 < \cdots < t_n$, $N(t_1) - N(t_0), \ldots, N(t_n) - N(t_{n-1})$ are independent.

Defining $o(\delta)$ as $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$, a Poisson process possess the following properties:

Property

- $N(t)=N(0+t)-N(0)\sim Poi(\lambda t)$
- For some $\delta\to 0$,
  - $P(N(\delta)=0)=1-\lambda\delta+o(\delta)$
  - $P(N(\delta)=1)=\lambda\delta+o(\delta)$
  - $P(N(\delta)=2)=o(\delta)$
- The autocovariance function is $\gamma(t_1,t_2)=\lambda\min\{t_1,t_2\}$

Construction

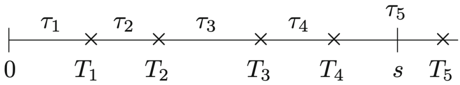
Let $\tau_1, \tau_2, \ldots$ be independent $\exp(\lambda)$ random variables. Let $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$. Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean λs.

Example: Customer arrivals

Let τ_n be the time between arrivals of customers, so that $T_n = \tau_1 + \cdots + \tau_n$ is the arrival time of the n-th customer. Let $N(s)$ be the number of arrivals by time s. Then, $N(s)$ follows a Poisson process with mean λs. For example,



Then, $N(s) = 4$ when $T_4 \leq s < T_5$.