Applied Stochastic Process

3b Renewal Process

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Let $\tau_1, \tau_2, \ldots$ be independent $\exp(\lambda)$ random variables. Let $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$. Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean λs.

Let $X_1, X_2, \ldots$ be independent random variables with some distribution F. Let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Then,

$$N(s) = \max\{n : S_n < s\}$$

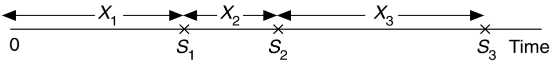
is a *renewal* process.

Example

Suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t), t \geq 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t.

Distribution of $N(t)$

Notice that the number of renewals by time t is greater than or equal to n if and only if the n-th renewal occurs before or at time t, i.e., $N(t) \geq n \iff S_n \leq t$,



Therefore, the distribution of $N(t)$ is related to that of S_n by

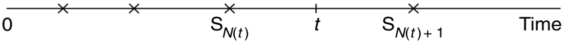
$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where F_n is the distribution of $S_n = \sum_{i=1}^{n} X_i$.

Limit theorem

Notice that:

- $S_{N(t)}$ represents the time of the last renewal *prior to or at* time t; and that
- $S_{N(t)+1}$ represents the time of the first renewal *after* time t,



We can show that, with probaility 1, as $t \to \infty$

$$\frac{N(t)}{t} \to \frac{1}{\mu}, \text{ where } \mu = \mathbb{E}\left[X_i\right].$$

The number $1/\mu$ is called the *rate* of the renewal process.

- GI (General input): Time between successive arrivals are independent with distribution F and mean $1/\lambda$.
- G (General service time): The i-th customer requires an amount of service s_i, which is independent with distribution G and mean $1/\mu$.
- 1 (One server)

Theorem

Suppose $\lambda < \mu$. If the queue starts with some finite number $k \geq 1$ customers who need service, then it will empty out with probability one. That is, the queue is stable. Furthermore, the limiting fraction of time the server is busy is at most λ/μ.