Applied Stochastic Process

4a Random Walk

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Random Walk

Recall the example of gambler's ruin. Now suppose that it is a fair game, i.e., one earns or lose $1 with probability 0.5, and that one can borrow any amount of money at no cost. Let x_0 be the initial amount of one's fortune, then one's fortune at time t can be written as

$$x_t = x_{t-1} + \varepsilon_t, \qquad t = 1, 2, \ldots$$

We call it a *random walk.*

Random Walk

Let ε_t be a white noise process with zero mean and variance σ^2, or $\varepsilon_t \sim WN(0, \sigma^2)$, i.e., for all s, t,

$$\mathbb{E}[\varepsilon_t] = 0, \qquad \operatorname{cov}(\varepsilon_s, \varepsilon_t) = \begin{cases} 0, & s \neq t \\ \sigma^2, & s = t \end{cases}$$

The cumulation of ε_t is called a random walk,

$$x_t = x_{t-1} + \varepsilon_t = x_0 + \sum_{s=1}^{t} \varepsilon_s, \qquad t = 1, 2, \ldots$$

Since ε_t is a white noise process, the moments of x_t can be obtained easily:

$$\mu = \mathbb{E}\left[x_t\right] = \mathbb{E}\left[x_0 + \sum_{s=1}^{t} \varepsilon_s\right] = x_0$$

$$\sigma_t^2 = \text{var}(x_t) = \text{var}\left(x_0 + \sum_{s=1}^{t} \varepsilon_s\right) = t\sigma^2$$

Since σ_t^2 increases with t, x_t is not stationary.

From Random Walk to Brownian Motion

Define an equidistant, disjoint partition of the continuous time interval $[0,1]$

$$[0,1) = \bigcup_{i=1}^{n} \left[\frac{i-1}{n}, \frac{i}{n}\right)$$

Interval-by-interval, define a scaled random walk as a continuous-time process of step function:

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{i-1} \varepsilon_j \qquad \text{for } t \in \left[\frac{i-1}{n}, \frac{i}{n}\right), \; i = 1, \ldots, n.$$

Define in addition for $t = 0$ and 1

$$X_n(0) = 0, \qquad X_n(1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varepsilon_j.$$

Suppose that $\sigma^2 = 1$, then by the Central Limit Theorem,

$$X_n(1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varepsilon_j \xrightarrow{d} \mathcal{N}(0,1).$$

Similarly, for any fixed $t \in (0,1)$, let $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x,

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \varepsilon_j = \frac{\sqrt{t}}{\sqrt{nt}} \sum_{j=1}^{\lfloor tn \rfloor} \varepsilon_j \xrightarrow{d} \mathcal{N}(0,t).$$

Theorem

Let $S_n = \sum_{i=1}^{n} \varepsilon_i,\ \varepsilon_i \overset{iid}{\sim} (0,1)$ *be a random walk. Define the rescaled partial-sum process*

$$X_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1].$$

Then, the Donsker's Theorem, or the functional central limit theorem, states that $X_n(t) \Rightarrow W(t), t \in [0,1]$*, where* $W(t)$ *is the Wiener process.*

Donsker's Theorem