Applied Stochastic Process

4b Brownian Motion

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Standard Brownian Motion

History

- In 1827, the botanist Robert Brown observed pollen grains suspended in water and described the erratic and continuous movement of tiny particles ejected from the grains.
- In 1905, Albert Einstein explained the movement by the continual bombardment of the immersed particles by the molecules in the liquid.
- The mathematical properties of the trajetory of a single particle is studied by the mathematician Norbert Wiener.

Einstein showed that the position x of a particle at time t was described by the partial differential heat equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t),$$

which solution is given by

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-x^2/2t},$$

which is the probability density function of the normal distribution with mean 0 and variance t

Definition

A stochastic process $W(t),\ t \in [0,T]$ is said to be a *Wiener process*, or a *standard Brownian motion*, if:

1. Zero starting value: $P(W(0)=0)=1$;
2. Independent increments: for any $0 \le t_0 \le t_1 \le \cdots \le t_n$, $W(t_1)-W(t_0), \ldots, W(t_n)-W(t_{n-1})$ are independent;
3. Stationary increments: $W(t+s)-W(s) \sim \mathcal{N}(0,t)$ for any $s,t>0$.

Properties of a Wiener process:

- **Moments**: For each s, t, $W(t) \sim \mathcal{N}(0, t)$, and $\text{cov}(W(s), W(t)) = \min\{s, t\}$.
- **Sample path**: For each realization, $W(t)$ is everywhere continuous but nowhere differentiable.
- **Markovian**: $P(W(t+s) \leq w | \mathcal{I}_t) = P(W(t+s) \leq w | W(t))$.
- **Martingale**: $\mathbb{E}\left[|W(t)|\right] < \infty$ and $\mathbb{E}\left[W(t+s) | \mathcal{I}_t\right] = W(t)$.
- **Scale invariance/self-similar**: The distributions of $W(\sigma t)$ and $\sqrt{\sigma} W(t)$ are the same.

Theorem

Let T_a denote the first time that the Wiener process hits $a > 0$. Then, the distribution of T_a is given by

$$\begin{aligned} P(T_a \leq t) &= 2P(X(t) \geq a) \\ &= 2P(X(1) \geq a/\sqrt{t}) \\ &= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} \mathrm{e}^{-x^2/2} dx \end{aligned}$$

Moreover,

1. $P(T_a > t) \to 0$ *as* $t \to \infty$.
2. $\mathbb{E}[T_a]$ *does not exist.*

Variations on Brownian Motion

Suppose $\epsilon_t \overset{\text{iid}}{\sim} (\mu, \sigma^2)$, consider the process

$$
\begin{aligned}
x_t &= x_{t-1} + \epsilon_t \\
&= x_0 + \sum_{s=1}^{t} \epsilon_s \\
&= x_0 + t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s
\end{aligned}
$$

where $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$. We call x_t a random walk with drift.

Definition

For any real μ and $\sigma > 0$, let B_t denotes the standard Brownian motion. The process

$$X_t = \mu t + \sigma B_t, \qquad t \geq 0$$

is called Brownian motion with drift parameter μ and variance parameter σ^2.

Example: Home team advantage (Stern, 1994)

For $0 \leq t \leq 1$, let X_t denote the difference in scores between the home and visiting teams after $100t$ percent of the game has been completed. The process is modeled as a Brownian motion with drift parameter μ being a measure of home team advantage.

1. What is the probability that the home team wins the game, given that they have an l points lead at time $t < 1$?
2. Stern (1994) estimated that $\widehat{\mu} = 4.87$ and $\widehat{\sigma} = 15.82$ with the results of 493 NBA games in 1992. What is the probability that the home team combacks from five points in the last five minutes of play ($t = 0.9$)?

Example: Home team advantage (Stern, 1994)

	Lead						
Time t	$l = -10$	$l = -5$	$l = -2$	$l = 0$	$l = 2$	$l = 5$	$l = 10$
0.00				0.62			
0.25	0.32	0.46	0.55	0.61	0.66	0.74	0.84
0.50	0.25	0.41	0.52	0.59	0.65	0.75	0.87
0.75	0.13	0.32	0.46	0.56	0.66	0.78	0.92
0.90	0.03	0.18	0.38	0.54	0.69	0.86	0.98
1.00	0.00	0.00	0.00		1.00	1.00	1.00

Table: Probabilities $p(l,t)$ that the Home team wins the game given that they are in the lead by l points after a fraction t of the game is completed

Definition

Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift parameter μ and variance parameter σ^2. The process $(G_t)_{t\geq 0}$

$$G_t = G_0 \mathrm{e}^{X_t}, \qquad t \geq 0,\ G_0 > 0$$

is called geometric Brownian motion.

Taking logarithm of G_t, $\ln G_t = \ln G_0 + X_t$. Therefore,

$$\ln G_t \sim \mathcal{N}(\ln G_0 + \mu t, \sigma^2 t).$$

G_t is said to have a lognormal distribution, with moments

$$\mathbb{E}\left[G_t\right] = G_0 \mathrm{e}^{(\mu+\sigma^2/2)t}, \qquad \mathrm{var}(G_t) = G_0^2 \mathrm{e}^{2t(\mu+\sigma^2/2)}(\mathrm{e}^{t\sigma^2} - 1)$$

On average, geometric Brownian motion exhibits exponential growth with growth rate $\mu + \sigma^2/2$.