Chapter 1 - Poisson Process

Applied Stochastic Process

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1 Poisson process

This section considers the Poisson process, which can be defined as follows.

Definition 1. $\{N(t), t \geq 0\}$ is a Poisson process if

1. $N(0) = 0$
2. $N(t)$ has *stationary* increments, specifically, $N(t+s) - N(s) \sim Poi(\lambda t)$
3. $N(t)$ has *independent* increments, i.e., for $t_0 < t_1 < \cdots < t_n$, $N(t_1) - N(t_0), \ldots, N(t_n) - N(t_{n-1})$ are independent.

In the following, we will introduce two ways to construct a Poisson process.

1.1 Construction method I

1.1.1 Poisson distribution

We first begin with the definition of the Poisson distribution. Consider a sequence of random variables $\{X_n(i)\}$ with independent Bernoulli distribution and success probability

$$P(X_n(i) = 1) = p_n = \lambda/n.$$

Then we can define another sequence of random variables $\{N_n\}$ such that

$$N_n = \sum_{i=1}^{n} X_n(i) \sim Bin(n, p_n).$$

The distribution function of N_n is given by

$$P(N_n = j) = C_n^j p_n^j (1 - p_n)^{n-j}.$$

The two distributions can be interpreted as follows:

- $X_n(i)$: whether there is any customer entering a shop within the period $(i-1)/n \leq t < i/n$;
- N_n: the number of time intervals (e.g. hour) that at least one customer enters a shop.

Time is discretized in the above two distributions. When $n \to \infty$, the time interval becomes very short and at the limit we have continuous time. In this case, we can show that

$$\begin{aligned}
\lim_{n\to\infty} P(N_n = j) &= \lim_{n\to\infty} C_n^j p_n^j (1-p_n)^{n-j} \\
&= \lim_{n\to\infty} \frac{n!}{j!(n-j)!} \frac{\lambda^j}{n^j} \left(1 - \frac{\lambda}{n}\right)^{n-j} \\
&= \lim_{n\to\infty} \underbrace{\frac{n}{n} \frac{n-1}{n} \cdots \frac{n-j+1}{n} \cdot \frac{1}{\left(1-\frac{\lambda}{n}\right)^j}}_{\to 1} \cdot \frac{\lambda^j}{j!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to \mathrm{e}^{-\lambda}} \\
&= \mathrm{e}^{-\lambda} \frac{\lambda^j}{j!}.
\end{aligned}$$

Let $N = \lim_{n\to\infty} N_n$, then we say N follows the Poisson distribution $Poi(\lambda)$, where $\mathbb{E}[N] = \lambda$. It can be interpreted as

- N: the number of customers entering a shop in one unit of time.

1.1.2 From Poisson distribution to Poisson process

The above only considers the situation of *one* unit of time. What if we are observing for *two* units of time instead? Again, we begin with discrete time. In 2 units of time, there are $2n$ independent trials. Therefore,

$$N_n(2) = \sum_{i=1}^{2n} X_n(i) \sim Bin(2n, \lambda/n).$$

Since now we have two times more independent trials of the same event, the expectation of $N_n(2)$ also doubles, $\mathbb{E}[N_n(2)] = 2\lambda$. As $n \to \infty$, the binomial distribution again converges to the Poisson distribution, with mean 2λ, i.e.,

$$N(2) = \lim_{n\to\infty} N_n(2) \sim Poi(2\lambda).$$

More generally, for any arbitrary $t > 0$, and let $\lfloor tn \rfloor$ be the largest integer smaller than or equal to tn,

$$N_n(t) = \sum_{i=1}^{\lfloor tn \rfloor} X_n(i) \sim Bin(\lfloor tn \rfloor, \lambda/n).$$

As $n \to \infty$,

$$N(t) = \lim_{n\to\infty} N_n(t) \sim Poi(t\lambda).$$

We call $N(t)$ a *Poisson process.*

Definition 2. Define a double sequence $\{X_n(i), i = 1, \ldots, n\}$ such that $X_n(i)$ follows independent Bernoulli distributions with probability distribution function $P(X_n(i) = 1) = p_n = \lambda/n$ and $P(X_n(i) = 0) = 1 - p_n$. Then the sequence of random variables $\{N(t), t \leq 0\}$ is said to be a Poisson process if

$$N(t) = \lim_{n\to\infty} \sum_{i=1}^{\lfloor tn \rfloor} X_n(i), \qquad N(0) = X_n(0) = 0.$$

We can now verify that the three definitions of a Poisson process in Definition 1 are satisfied.

1. $N(0) = 0$: This one holds by definition.

2. *Stationary increments*: First, notice that

$$N(t+s) - N(s) = \lim_{n\to\infty} \sum_{i=1}^{\lfloor (t+s)n \rfloor} X_n(i) - \lim_{n\to\infty} \sum_{i=1}^{\lfloor sn \rfloor} X_n(i) = \lim_{n\to\infty} \sum_{i=\lfloor sn \rfloor+1}^{\lfloor (t+s)n \rfloor} X_n(i).$$

   Therefore, $N(t+s) - N(s)$ consists of the sum of tn independent and identical Bernoulli random variables. As $n \to \infty$, it converges to $Poi(t\lambda)$. Since it does not depend on s, we say it has stationary increments.

3. *Independent increments*:Consider two non-overlapping interval (t_1, t_2) and (t_3, t_4), such that $t_1 < t_2 \leq t_3 < t_4$. Then,

$$N(t_2) - N(t_1) = \lim_{n\to\infty} \sum_{i=\lfloor t_1 n \rfloor+1}^{\lfloor t_2 n \rfloor} X_n(i),$$
$$N(t_4) - N(t_3) = \lim_{n\to\infty} \sum_{i=\lfloor t_3 n \rfloor+1}^{\lfloor t_4 n \rfloor} X_n(i).$$

   Since $X_n(i)$ in $i \in \{\lfloor t_1 n \rfloor + 1, \ldots, \lfloor t_2 n \rfloor\}$ are all independent of $X_n(j)$ in $j \in \{\lfloor t_3 n \rfloor + 1, \ldots, \lfloor t_4 n \rfloor\}$, $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are also independent.

1.2 Construction method II

1.2.1 Exponential distribution

We again begin with the discrete time. Consider the same sequence of Bernoulli random variables $\{X_n(i)\}$, and suppose now we want to know how many independent trials T_n are needed before the first occurrence of an event. Then, T_n is said to follow a geometric distribution with probability distribution function

$$P(T_n = j) = p_n(1-p_n)^j.$$

We can also derive the cumulative distribution function

$$P(T_n \leq j) = \sum_{k=0}^{j} p_n(1-p_n)^k = p_n \frac{1-(1-p_n)^{j+1}}{p_n} = 1 - (1-p_n)^{j+1}$$

Now let $\tau_n = T_n/n$ and $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} P(\tau_n \leq t) &= \lim_{n\to\infty} P(T_n/n \leq t) = \lim_{n\to\infty} P(T_n \leq nt) \\
&= 1 - \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{\lfloor nt \rfloor + 1} = 1 - \lim_{n\to\infty} \left[\left(1 - \frac{\lambda}{n}\right)^n\right]^{\frac{\lfloor nt \rfloor}{n} + \frac{1}{n}} \\
&= 1 - \mathrm{e}^{-\lambda t}.
\end{aligned}$$

We say $\tau = \lim_n \tau_n$ has an exponential distribution with rate λ. The two distributions can be interpreted as follows:

- T_n: the number of time intervals (e.g. hour) until the first customer enters a shop;
- τ: the length of time until the first customer enters a shop.

The only difference between the geometric distribution and the exponential distribution is whether we treat time as discrete values or a continuous value. A very important feature of the two distributions is their *lack of memory*, i.e.,

$$P(\tau > t + s | \tau > t) = P(\tau > s).$$

Recall that the fundamental elements in the construction of these two distributions are the independent Bernoulli trials $X_n(i)$. The condition $\tau > t$ is equivalent to saying $X_n(1) = X_n(\lfloor nt \rfloor) = 0$. However, it does not provide any information about $X_n(i), i \geq \lfloor nt \rfloor$, which are the components of $\tau | \tau > t$. Suppose we have already waited for 5 minutes and there is no customer at all. We now ask what the distribution function of the arrival time of the next customer is. If it follows an exponential distribution, then regardless of how long we have waited, the arrival time of the next customer still follows the same exponential distribution.

1.2.2 From exponential distribution to Poisson process

Instead of monitoring whether a customer shows up at every instance, we can alternatively ask the question "*when will the next customer arrive*". We therefore have another definition of Poisson process.

Definition 3. Let $\tau_1, \tau_2, \ldots$ be independent $\exp(\lambda)$ random variables. Let $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$. Then,

$$N(t) = \max\{n : T_n < t\}$$

is a Poisson process with mean λt.

In the above definition, T_n is the time for n customers to enter a shop. $N(t)$ is the maximum of n such that $t > T_n$, meaning that at time t, there is enough time for n customers to show up, but not enough for the $(n+1)$-th one. $N(t)$ equals n after the n-th customers enter the shop, and before the $(n+1)$-th one comes.

2 Renewal process

As discussed above, the *memorylesssness* of a Poisson process may not be desirable in many applications. Noticing that this feature comes from the memorylesssness of the exponential distribution, we can replace the exponential distribution by other (positive) distributions in the definition of the Poisson process.

Definition 4. Let $X_1, X_2, \ldots$ be independent random variables with some distribution F. Let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Then,

$$N(s) = \max\{n : S_n < s\}$$

is a *renewal* process.

Example 1. Suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t), t \geq 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t.

In the definition of a renewal process, assumptions are only made on the arrival time X_i. Nothing is said about the distribution of the process $N(t)$ itself. To find the distribution of $N(t)$, we first have to notice that the following two sentences are equivalent:

- $N(t) \geq n$: At time t, at least n lightbulbs have broken.
- $S_n \leq t$: The n-th lightbulb broke before time t.

Therefore, the distribution of $N(t)$ is related to that of S_n by

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t), \end{aligned}$$

where F_n is the distribution of $S_n = \sum_{i=1}^{n} X_i$.

Theorem 2.1. *Let $N(t)$ be a renewal process with arrival time X_i, and let $\mathbb{E}[X_i] = \mu^{-1}$. Then, as $t \to \infty$, $N(t)/t \to \mu$. The number μ is called the rate of the renewal process.*

Proof. First, notice that:

- $S_{N(t)}$ represents the time of the last renewal *prior to or at* time t; and that
- $S_{N(t)+1}$ represents the time of the first renewal *after* time t.

Therefore,

$$S_{N(t)} \leq t \leq S_{N(t)+1} \iff \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}.$$

As $t \to \infty$ and so $N(t) \to \infty$, the left-hand side of the above inequality converges to

$$\frac{S_{N(t)}}{N(t)} = N(t)^{-1} \sum_{i=1}^{N(t)} X_i \to \mathbb{E}[X_i] = \frac{1}{\mu}.$$

Similarly, the right-hand side converges to

$$\frac{S_{N(t)+1}}{N(t)} = (N(t)+1)^{-1} \sum_{i=1}^{N(t)+1} X_i \cdot \frac{N(t)+1}{N(t)} \to \mathbb{E}[X_i] = \frac{1}{\mu}.$$

Therefore,

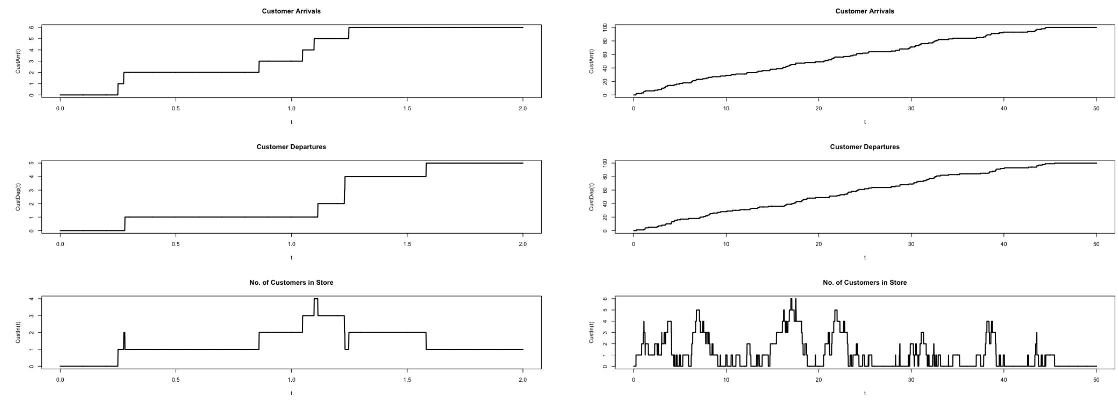
$$\frac{t}{N(t)} \to \frac{1}{\mu} \implies \frac{N(t)}{t} \to \mu.$$

□

3 $GI/G/1$ Queue

An important application of the renewal process is in queuing theory. Queues can be found in many places, e.g., the service desk in a bank, the highway toll station, etc. In a queue, there are two types of people: input (e.g., customers) and server (e.g., the bank teller). A $GI/G/1$ Queue can be characterized as follows:

- GI (General input): Time between successive arrivals are independent with distribution F and mean $1/\lambda$.
- G (General service time): The i-th customer requires an amount of service s_i, which is independent with distribution G and mean $1/\mu$.
- 1 (One server)



The two figures above plot the simulated customer arrival (top), departure (middle) and number of customers in store (bottom). We observe that the number of customers in store (queue) piles up when the service time is longer than the arrival time of new customers. When we simulate the queue for a long time (right), we observe that the queue always empty out.

Theorem 3.1. *Suppose $\lambda < \mu$. If the queue starts with some finite number $k \geq 1$ customers who need service, then it will empty out with probability one. That is, the queue is stable. Furthermore, the limiting fraction of time the server is busy is at most λ/μ.*

Proof. The first part is obvious. For the second part, the busy time up to T_n (the time when the n-th customer enters the shop) is at most $Z_0 + s_1 + \cdots + s_{n-1}$, where Z_0 is the service time for the k customers in the initial queue. Since the n-th customer just arrives at time T_n, the server can at most serve up to the $(n-1)$-th customer. The fraction of busy time is at most

$$\frac{Z_0 + s_1 + \cdots + s_{n-1}}{T_n} = \frac{\frac{Z_0}{n} + \frac{s_1+\cdots+s_{n-1}}{n-1}\frac{n-1}{n}}{\frac{t_1+\dot{+}t_n}{n}} \to \frac{1/\mu}{1/\lambda} = \frac{\lambda}{\mu}.$$

□