



# Applied Stochastic Process

## 1 Poisson Process and Renewal Process

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# Poisson and Exponential Distributions

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Suppose  $X_n \sim \text{Bin}(n, \lambda/n)$ . Let  $X = \lim_{n \rightarrow \infty} X_n$ , when  $n \rightarrow \infty$  and  $p \rightarrow 0$ , we can show that

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, \quad i = 0, 1, \dots$$

We say  $X$  has the Poisson distribution with parameter  $\lambda$ , or  $X \sim \text{Poi}(\lambda)$ .

Some properties of the Poisson distribution:

- ▶ Moments:  $\mathbb{E}[X] = \lambda$ ,  $\text{var}(X) = \lambda$ .
- ▶ Independent sums: If  $X_i$  are independent  $Poi(\lambda_i)$ , then  $X_1 + \cdots + X_n \sim Poi(\lambda_1 + \cdots + \lambda_n)$

Suppose  $T_n \sim \text{Geo}(p_n)$  where  $p_n = \lambda/n$  and  $T = \lim_{n \rightarrow \infty} n^{-1}T_n$ , then one can show that

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0.$$

We say  $T$  is exponentially distributed with parameter  $\lambda$ , or  $T \sim \text{exp}(\lambda)$ . Its probability density function is given by

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Some properties of the exponential distribution:

- ▶ Moments:  $\mathbb{E}[T] = 1/\lambda$ ,  $\text{var}(T) = 1/\lambda^2$ .
- ▶ Lack of memory:  $P(T > t + s | T > t) = P(T > s)$ .
- ▶ Exponential races: If  $S \sim \text{exp}(\lambda)$  and  $T \sim \text{exp}(\mu)$  are independent, then  $\min\{S, T\} \sim \text{exp}(\lambda + \mu)$ .

# Poisson Process

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## Definition

$\{N(t), t \geq 0\}$  is a Poisson process if

1.  $N(0) = 0$
2.  $N(t)$  has *stationary* increments, specifically,  
 $N(t+s) - N(s) \sim Poi(\lambda t)$
3.  $N(t)$  has *independent* increments, i.e., for  
 $t_0 < t_1 < \dots < t_n$ ,  $N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$  are independent.



Defining  $o(\delta)$  as  $\lim_{\delta \rightarrow 0} \frac{o(\delta)}{\delta} = 0$ , a Poisson process possess the following properties:

## Property

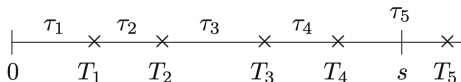
- ▶  $N(t) = N(0 + t) - N(0) \sim Poi(\lambda t)$
- ▶ For some  $\delta \rightarrow 0$ ,
  - ▶  $P(N(\delta) = 0) = 1 - \lambda\delta + o(\delta)$
  - ▶  $P(N(\delta) = 1) = \lambda\delta + o(\delta)$
  - ▶  $P(N(\delta) = 2) = o(\delta)$
- ▶ The autocovariance function is  $\gamma(t_1, t_2) = \lambda \min\{t_1, t_2\}$

Let  $\tau_1, \tau_2, \dots$  be independent  $\exp(\lambda)$  random variables. Let  $T_n = \tau_1 + \dots + \tau_n$ ,  $T_0 = 0$ . Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean  $\lambda s$ .

Let  $\tau_n$  be the time between arrivals of customers, so that  $T_n = \tau_1 + \cdots + \tau_n$  is the arrival time of the  $n$ -th customer. Let  $N(s)$  be the number of arrivals by time  $s$ . Then,  $N(s)$  follows a Poisson process with mean  $\lambda s$ . For example,



Then,  $N(s) = 4$  when  $T_4 \leq s < T_5$ .

## Renewal Process

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Let  $\tau_1, \tau_2, \dots$  be independent  $\exp(\lambda)$  random variables. Let  $T_n = \tau_1 + \dots + \tau_n$ ,  $T_0 = 0$ . Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean  $\lambda s$ .

Let  $X_1, X_2, \dots$  be independent random variables with some distribution  $F$ . Let  $S_n = X_1 + \dots + X_n$ ,  $S_0 = 0$ . Then,

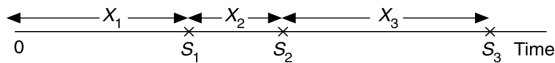
$$N(s) = \max\{n : S_n < s\}$$

is a *renewal* process.

## Example

Suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions,  $\{N(t), t \geq 0\}$  is a renewal process when  $N(t)$  represents the number of lightbulbs that have failed by time  $t$ .

Notice that the number of renewals by time  $t$  is greater than or equal to  $n$  if and only if the  $n$ -th renewal occurs before or at time  $t$ , i.e.,  $N(t) \geq n \iff S_n \leq t$ ,



Therefore, the distribution of  $N(t)$  is related to that of  $S_n$  by

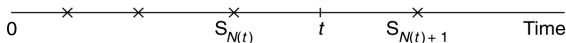
$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n + 1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where  $F_n$  is the distribution of  $S_n = \sum_{i=1}^n X_i$ .



Notice that:

- ▶  $S_{N(t)}$  represents the time of the last renewal *prior to or at* time  $t$ ; and that
- ▶  $S_{N(t)+1}$  represents the time of the first renewal *after* time  $t$ ,

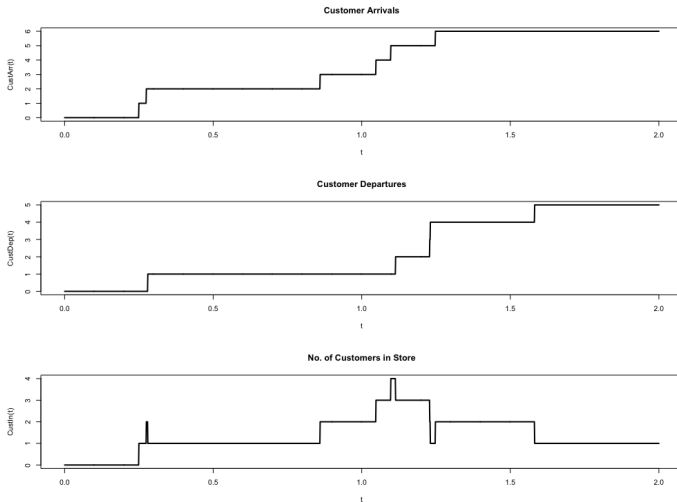


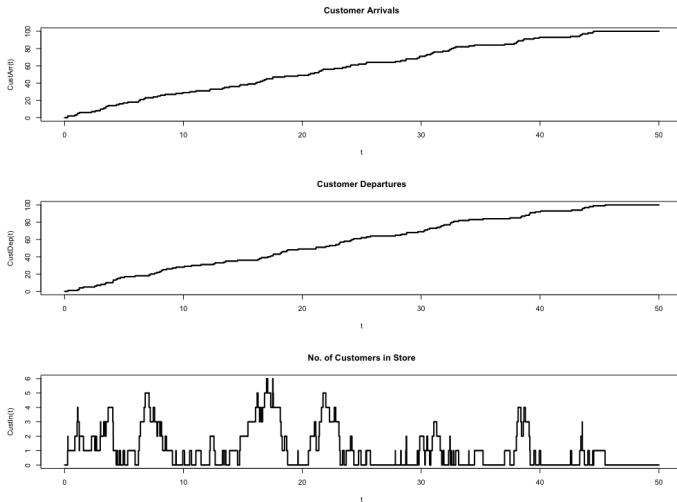
We can show that, with probability 1, as  $t \rightarrow \infty$

$$\frac{N(t)}{t} \rightarrow \mu, \text{ where } \mathbb{E}[X_i] = \frac{1}{\mu}.$$

The number  $\mu$  is called the *rate* of the renewal process.

- ▶  $GI$  (General input): Time between successive arrivals are independent with distribution  $F$  and mean  $1/\lambda$ .
- ▶  $G$  (General service time): The  $i$ -th customer requires an amount of service  $s_i$ , which is independent with distribution  $G$  and mean  $1/\mu$ .
- ▶ 1 (One server)





## Theorem

*Suppose  $\lambda < \mu$ . If the queue starts with some finite number  $k \geq 1$  customers who need service, then it will empty out with probability one. That is, the queue is stable. Furthermore, the limiting fraction of time the server is busy is at most  $\lambda/\mu$ .*