Applied Stochastic Process

2 Markov Chain

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Stochastic Process

	Discrete state	Continuous state
Discrete time	Markov chain	Random walk
Continuous time	Poisson process	Brownian motion

Markov Chain

Definition

A stochastic process $\{X(t), t \in \mathcal{T}\}$ is said to be Markovian if

$$P(X(t_n) \in A | X(t), t \leq t_{n-1}) = P(X(t_n) \in A | X(t_{n-1}))$$

where $t_{n-1} < t_n$.

Definition

A Markov chain with state space $\mathcal{X}$ is a sequence $X_0, X_1, \dots$ of $\mathcal{X}$-valued random variables such that for all states $i, j, k_0, k_1, \dots$ and all times $t = 0, 1, 2, \dots,$

$$\begin{aligned} P(X_{t+1} = j | X_t = i, X_{t-1} = k_{t-1}, \dots) &= P(X_{t+1} = j | X_t = i) \\ &= p_{i,j} \end{aligned}$$

where $p_{i,j}$ depends only on the states i, j and not on the time t or the previous states $k_{t-1}, k_{t-2}, \dots$.

Example: Gambler's Ruin

Example

Consider a gambling game in which on any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. Suppose initially you have \$2 and you will quit the game either if your fortune reaches \$4 or \$0. Let X_n be the amount you have after n plays.

1. What is the state space?
2. Is X_n Markovian?

The matrix $\mathbf{P}$ that collects the transition probabilities $p_{i,j}$ is called the *transition matrix*, defined as

$$\mathbf{P} = (p_{i,j})_{i,j} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,M} \\ \vdots & \ddots & \vdots \\ p_{M,1} & \cdots & p_{M,M} \end{pmatrix}$$

If the initial state is random with probability $\mathbf{q}_0 = (q_{01}, \ldots, q_{0M})$, then the unconditional probability at period 1 is given by $\mathbf{q}_1 = \mathbf{q}_0\mathbf{P}$.

Let $p_{ij}^{n} = P(X_{n+k} = j | X_k = i)$ be the n-step transition probabilities. Then the Chapman-Kolmogorov equations are

$$p_{ij}^{(n+m)} = \sum_{k=1}^{M} p_{ik}^{(n)} p_{kj}^{(m)} \qquad \text{for all } n, m \geq 0, \text{ all } i, j.$$

Moreover, let $\mathbf{P}^{(n)}$ be the matrix of n-step transition probabilities, then

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}.$$

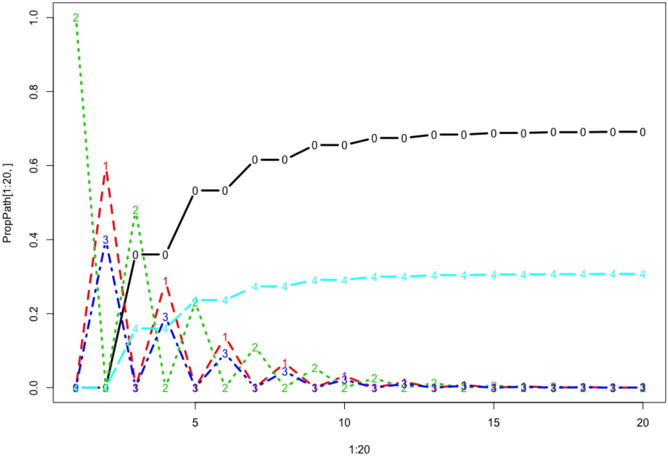
If the initial state is random with probability $\mathbf{q}_0$, then the unconditional probability at period n is given by $\mathbf{q}_n = \mathbf{q}_0 \mathbf{P}^n$.

Example

Consider a gambling game in which on any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. Suppose initially you have \$2 and you will quit the game either if your fortune reaches \$4 or \$0. Let X_n be the amount you have after n plays.

1. What is the transition matrix?
2. What is the probability that a player loses all the money?
3. On average, how long does one play before quitting the game?

Example: Gambler's Ruin (continued)

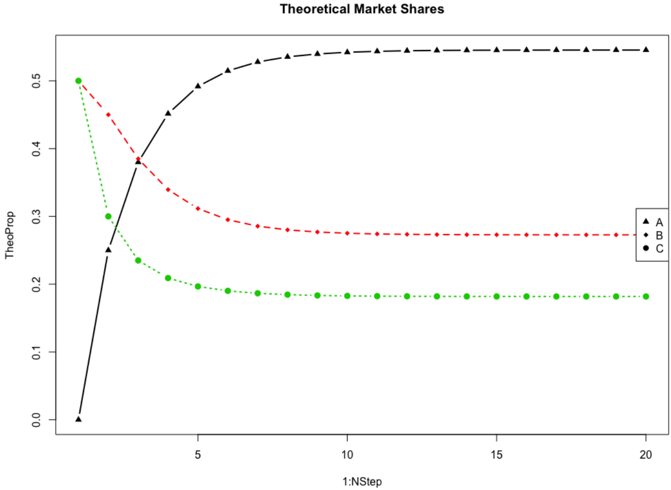


Example

Suppose there are three types of laundry detergent, and let X_n be the brand chosen on the nth purchase. Customers who try these brands are satisfied and choose the same thing again with probability 0.8, 0.6, and 0.4 respectively. When they change they pick one of the other two brands at random. Suppose also that $\mathbf{q}_0 = (0, 0.5, 0.5)$.

1. What is the transition probability?
2. Do the market shares of the three product stabilize?

Example: Brand Preference



Theorem

The limiting probabilities of a Markov chain always exist and will not depend on the initial state if the chain is aperiodic and irreducible. Moreover, it is given by the solution of

$$\pi_j = \sum_i \pi_i P_{ij}, \qquad \sum_j \pi_j = 1$$

or in matrix form $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$,

Classification of States

Definition

1. State j is said to be *accessible* from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.
2. Two states i and j that are accessible to each other are said to *communicate*, and we write $i \leftrightarrow j$.
3. Two states that communicate are said to be in the same *class*.
4. The Markov chain is said to be *irreducible* if there is only one class, i.e., if all states communicate with each other.

Example

Example

How many classes of states are there in the examples of Gambler's Ruin and Brand Preference?

Property

- $i \leftrightarrow i$ for all i.
- If $i \leftrightarrow j$, then $j \leftrightarrow i$.
- If $i \leftrightarrow j$, and $j \leftrightarrow k$, then $i \leftrightarrow k$.
- Any two classes of states are either identical or disjoint.

Definition

For any state i, let $f_{i,j}$ be the probability that, starting in state i, the process will ever enter state j. State i is said to be *recurrent* if $f_{i,i} = 1$ and *transient* if $f_{i,i} < 1$.

Example

Which states are transient in the examples of Gambler's Ruin and Brand Preference?

Property

- If state i is recurrent, then
  - starting in state i, the process will reenter state i infinitely often.
  - $f_{i,j} = f_{j,j} = 1$ if $i \leftrightarrow j$. Therefore, recurrence is a class property.
- It is impossible to go from a recurrent to a transient state.
- If state i is transient, then starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $1/(1 - f_{i,i})$.
- In a finite state Markov chain, at least one class of state is recurrent.

Mean time spent in transient states

Theorem

Let $\mathbf{P}_T$ *specifies only the transition probabilities from transient states into transient states, and let* $\mathbf{S} = (s_{i,j})_{i,j=1,\ldots,t}$ *denote the matrix that collects the expected number of time periods that the Markov chain is in state* j*, given that it starts in state* i*. Then,*

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$$

where $\mathbf{I}$ *is the identity matrix.*

Probability of j given i

Theorem

The probability that the Markov chain ever makes a transition into a transient state j given that it starts in a transient state i is given by

$$f_{i,j} = \frac{s_{ij} - \delta_{i,j}}{s_{jj}}, \qquad \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Positive recurrent

Definition

If state j is recurrent, let m_j be the expected number of transitions that it takes the Markov chain when starting in state j to return to that state. Then, the recurrent state j is said to be *positive recurrent* if $m_j < \infty$ and *null recurrent* if $m_j = \infty$.

Property

Let π_j be the long run proportion of time in state j.

- If the Markov chain is irreducible and recurrent, then for any initial state $\pi_j = 1/m_j$.
- If i is positive recurrent and $i \leftrightarrow j$, then j is positive recurrent.
- Null recurrence is also a class property.
- An irreducible finite state Markov chain must be positive recurrent.

Theorem

The long-run proportions of an irreducible Markov chain are the unique solution of the equations

$$\pi_j = \sum_i \pi_i p_{ij}, \qquad \sum_j \pi_j = 1$$

or in matrix form $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, *if the chain is positive recurrent. Moreover, if there is no solution of the preceding linear equations, then the Markov chain is either transient or null recurrent and all* $\pi_j = 0$.

Periodic Markiv chain

Definition

The *period* of a state is the largest number that will divide all the $n \geq 1$ for which $p_{i,i}^{(n)} > 0$. A Markov chain is said to be *aperiodic* if all of its states have period one, and it is *periodic* otherwise.

Theorem

- *A periodic Markov chain does not have limiting probabilities.*
- *The limiting probabilities of an irreducible, aperiodic chain always exist and do not depend on the initial state.*
- *The limiting probabilities, when they exist, will equal the long-run proportions.*
- *An irreducible, positive recurrent, aperiodic Markov chain is said to be ergodic.*

Definition

The stochastic process $\{X(t), t \in \mathcal{T}\}$ is said to be *ergodic* if any characteristics of the process can be obtained, with probability 1, from a single realization.

Example

In the examples of Gambler's Ruin and Brand Preference, do the stationary distributions exist?