Applied Stochastic Process

2A Branching Processes

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Consider a population of reproducing individuals with the following features:

- Initial population: $X_0 = 1$
- Lifetime of each individual: 1 unit of time
- Number of offsprings: $j \leq 0$ with probability P_j, independent of the numbers produced by other individuals

The branching process is $\{X_0, X_1, \ldots\} = \{X_n : n \in \mathbb{N}\}$, given by

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}, \qquad P(Z_{n,i} = j) = P_j, \ j \geq 0$$

where $Z_{n,i}$ is the number of offspring produced by individual i at time n.

Spread of COVID-19 as a Branching Process

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Reporting, Epidemic Growth, and Reproduction Numbers for the 2019 Novel Coronavirus (2019-nCoV) Epidemic FREE

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Background: Virologically confirmed cases of 2019 novel coronavirus (2019-nCoV) in China and other countries have increased sharply (1, 2), leading to concerns regarding its pandemic potential. Viral epidemiology has been characterized sufficiently to permit construction of transmission models that predict the future course of this epidemic (3).

Objective: To provide insight into the changing nature of case findings and epidemic growth.

Methods: We developed a simple disease-transmission model in which the 2019-nCoV epidemic was modeled as a branching process starting in mid-November 2019, with a serial interval of 7 days (time between cases) and a basic reproduction number (R_0) of 2.3 (new cases from each old case), based on available data and assuming no intervention (Figure 1). The epidemic start date aligned our modeled case counts to point estimates from international case exportation data (4). The model estimated plausible values of the effective reproduction number (R_e; reproduction number in the presence of control efforts) after implementation of a quarantine in Wuhan and surrounding areas of China on 24 January 2020 (3) (Figure 1).

Some properties:

- State 0 is a recurrent state, since $P_{00} = 1$.
- If $P_0 > 0$, then all other states are transient.
- If $P_0 > 0$, then the population will either die out or its size will diverge to infinity.
- Every individual in every generation starts a new, independent branching process.

What is the transition probability?

Let μ and σ^2 be the mean and variance of the number of offspring produced by a single individual,

$$\mathbb{E}\left[Z_{n,i}\right] = \mu, \qquad \text{var}(Z_{n,i}) = \sigma^2.$$

Then, what are the exepcted value and variance of X_n?

- **Law of Iterated Expectation (LIE)**:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$$

- **Law of Total Variance (LTV)**:

$$\text{var}(Y) = \mathbb{E}[\text{var}(Y|X)] + \text{var}(\mathbb{E}[Y|X])$$

Expectation of X_n

Conditioning on X_{n-1},

$$\begin{aligned}
\mathbb{E}\left[X_n\right] &= \mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{i,n}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{i,n} | X_{n-1}\right]\right] \\
&= \mu \mathbb{E}\left[X_{n-1}\right] \\
&= \ldots \\
&= \mu^n \mathbb{E}\left[X_0\right] \\
&= \mu^n
\end{aligned}$$

Variance of X_n

Conditioning on X_{n-1},

$$\mathbb{E}\left[X_n | X_{n-1}\right] = X_{n-1}\mu, \qquad \operatorname{var}(X_n | X_{n-1}) = X_{n-1}\sigma^2$$

By the Law of Total Variance,

$$\begin{aligned}
\operatorname{var}(X_n) &= \mathbb{E}\left[\operatorname{var}(X_n | X_{n-1})\right] + \operatorname{var}(\mathbb{E}\left[X_n | X_{n-1}\right]) \\
&= \mathbb{E}\left[X_{n-1}\sigma^2\right] + \operatorname{var}(X_{n-1}\mu) \\
&= \sigma^2\mu^{n-1} + \mu^2 \operatorname{var}(X_{n-1})
\end{aligned}$$

By recursive substitution,

$$\operatorname{var}(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right), & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}$$

What is π_0?

What is the probability that the population will eventually die out, i.e., what is

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 | X_0 = 1)?$$

What is π_0?

1. If $\mu \leq 1$, then $\pi_0 = 1$.
2. If $\mu > 1$, then π_0 is the smallest positive number satisfying

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j.$$