Financial Modeling and Data Analysis

Portfolio Selection

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Outline

Risk-Return Trade-off

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Two principles of investment:

1. maximize the expected return;
2. minimize the risk (standard deviation of the return).

Usually, higher returns entail higher risks. Is there any optimal compromise between expected return and risk?

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Answer: Yes (In a mean-variance sense)!

One Risky Asset and One Risk-free Asset

Consider the following two assets:

- A risky asset with expected returns $\mathbb{E}[R] = \mu_1$ and variance $\text{var}(R) = \sigma_1^2$.
- A risk-free asset with a *fixed* return R_f.

If we invest a fraction w of our wealth in the risky asset and the remaining fraction $1 - w$ in the risk-free asset, what are the expected value and standard deviation of the portfolio return?

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Answer: $\mu_p = \mathbb{E}[R_p] = w\mu_1 + (1-w)R_f$, $\sigma_p = |w|\sigma_1$.

Example

Suppose the expected return of a risky asset is 15% and the standard deviation of the return is 25%. Assume also that the risk-free rate is 6%.

- If a portfolio is formed with $w = 0.6$ of the risky asset and $1 - w = 0.4$ of the risk-free asset, what are the expected value and standard deviation of the portfolio return?
- If the investor wants to fix the standard deviation of the return at 20%, how should you allocate the fund in the portfolio?
- Is it possible to achieve an expected return of 19.5%?

Solution

- *The expected value is* $\mu_p = wR_1 + (1-w)R_f = 11.4\%$ *and the standard deviation is* $\sigma_p = |w|\sigma_1 = 15\%$.
- $\sigma_p = |w|\sigma_1 = 0.25|w| = 0.2 \implies w = 0.8.$
- $\mu_p = wR_1 + (1-w)R_f = 0.15w + 0.06(1-w) = 0.195 \implies w = 1.5 > 1$. *Therefore, it is only possible if* ***short selling*** *is allowed.*

One risky asset and one risk-free asset

w_1	$1 - w_1$	μ_p	σ_p
1	1	6%	0%
0.5	0.5	10.5%	12.5%
1	0	15%	25%
1.5	-0.5	19.5%	37.5%
-0.5	1.5	1.5%	12.5%

Table: Portfolio risk and returns

Two Risky Assets

Suppose now that there are two risky assets:

- Asset 1: $\mathbb{E}[R_1] = \mu_1$ and $\text{var}(R_1) = \sigma_1^2$;
- Asset 2: $\mathbb{E}[R_2] = \mu_2$ and $\text{var}(R_2) = \sigma_2^2$;
- Correlation between the returns of the two assets $\text{corr}(R_1, R_2) = \rho_{12}$.

If we allocate w_1 and $w_2 = 1 - w_1$ of our wealth in the two risky assets respectively, what are the expected value and standard deviation of the portfolio returns?

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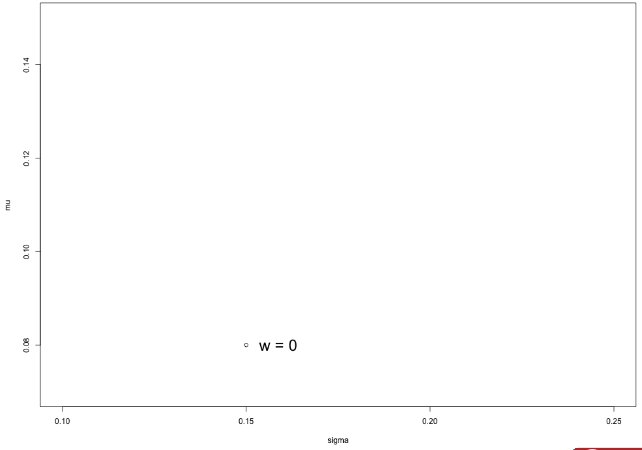
If we allocate w_1 and $w_2 = 1 - w_1$ of our wealth in the two risky assets respectively, what are the expected value and standard deviation of the portfolio returns?

Answer: $\mu_p = \mathbb{E}[R_p] = w_1\mu_1 + (1 - w_1)\mu_2$;
$\sigma_p = \sqrt{w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\rho_{12}\sigma_1\sigma_2}$.

Let $\mu_1 = 14\%$, $\mu_2 = 8\%$, $\sigma_1 = 20\%$, $\sigma_2 = 15\%$, $\rho_{12} = 0.2$.



Efficient frontier

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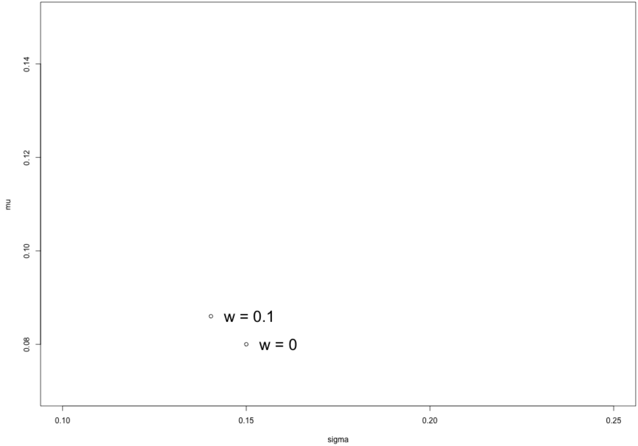

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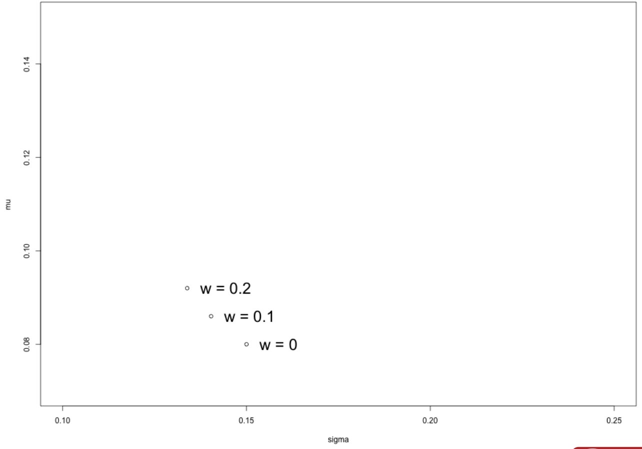

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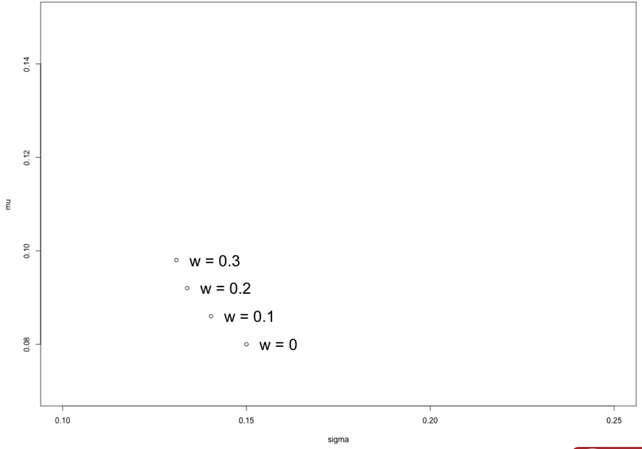

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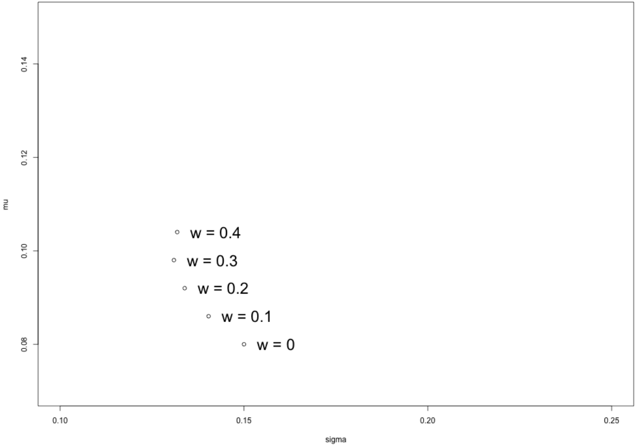

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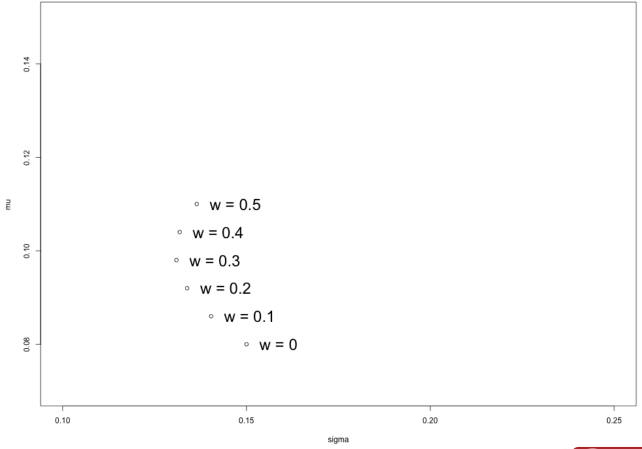

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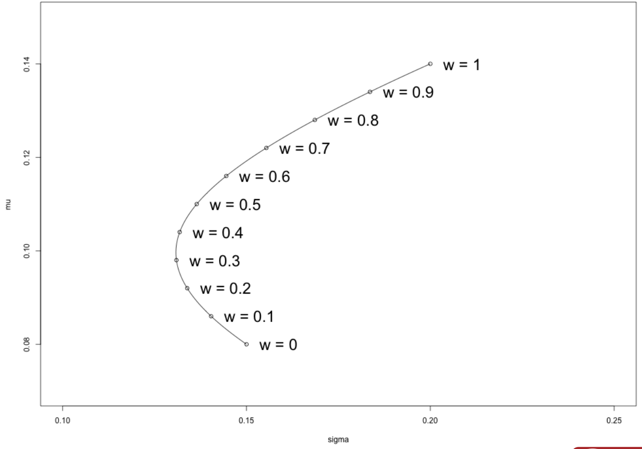

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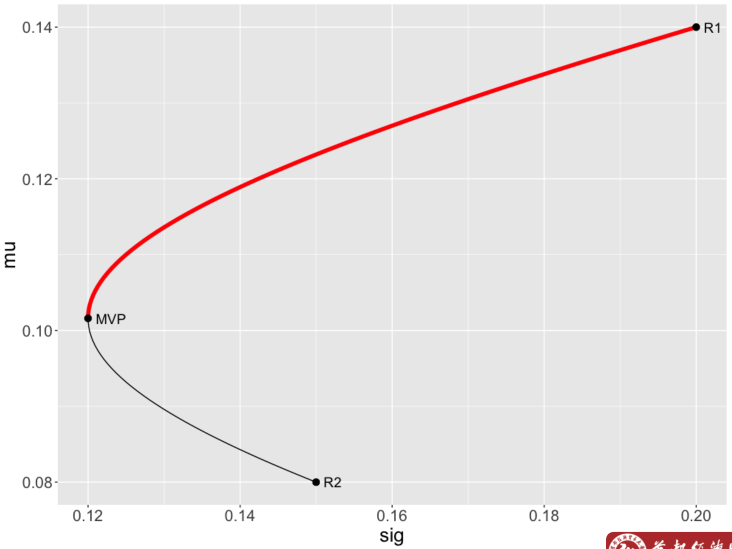

Exercise

Find w_1 of the *minimum variance portfolio*.

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Solution

We minimize σ_p^2 *with respect to* w_1*. Taking derivative of* σ_p^2 *with respect to* w_1*,*

$$\frac{\mathrm{d}\sigma_p^2}{\mathrm{d}w_1} = 2\sigma_1^2 w_1 - 2\sigma_2^2(1-w_1) + (2-4w_1)\rho_{12}\sigma_1\sigma_2 \overset{!}{=} 0.$$

Solving for w_1 *yields*

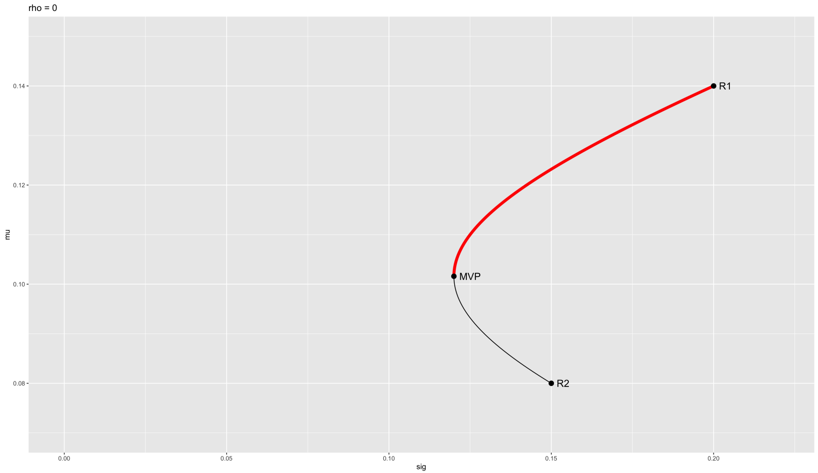
$$w_1^* = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}.$$

The role of ρ_{12}

- If $\rho_{12} > 0$, the two assets tend to move together which increases the volatility of the portfolio.
- If $\rho_{12} < 0$, a bad performance of one tends to occur with a good performance of the other, so the volatility of the portfolio decreases.

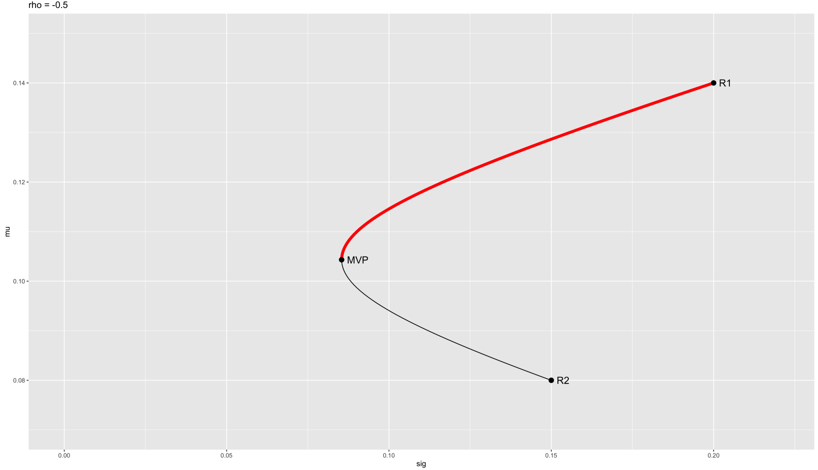
The role of ρ_{12}

$$\rho_{12} = 0$$



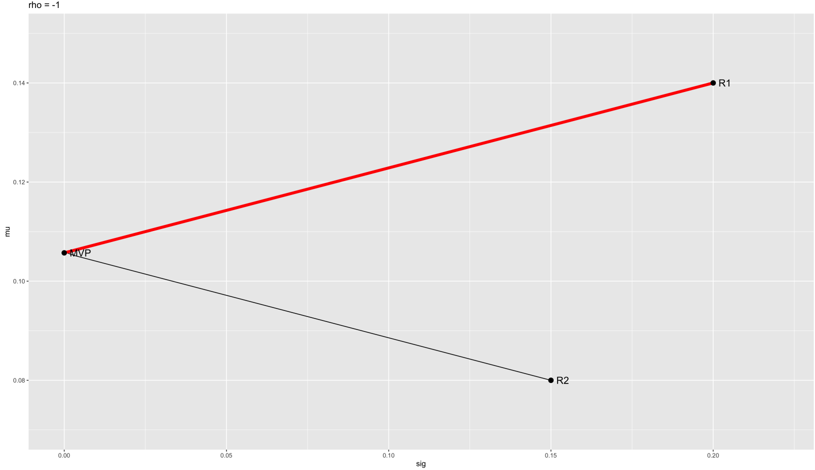
The role of ρ_{12}

$$\rho_{12} = -0.5$$

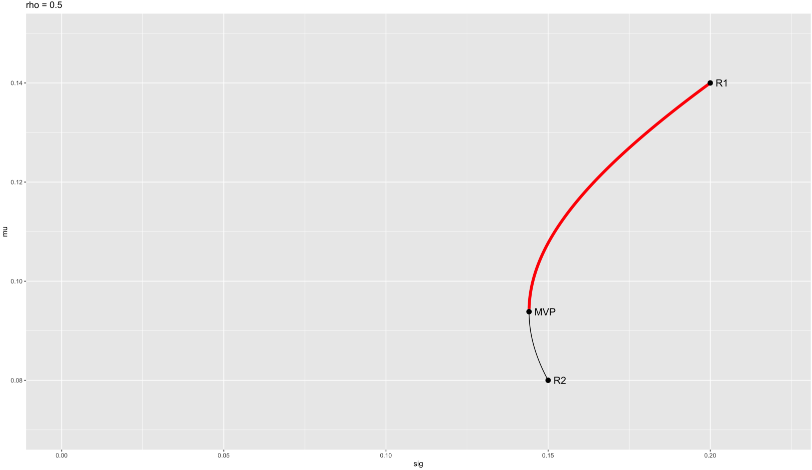


The role of ρ_{12}

$$\rho_{12} = -1$$

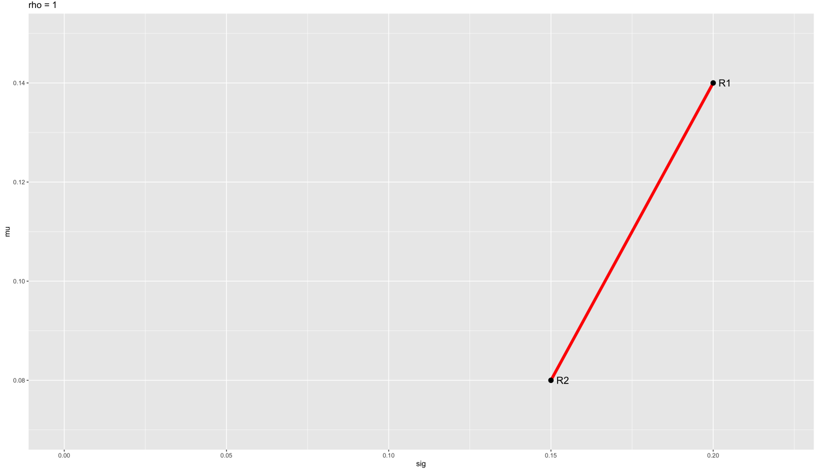


$\rho_{12} = 0.5$



The role of ρ_{12}

$$\rho_{12} = 1$$



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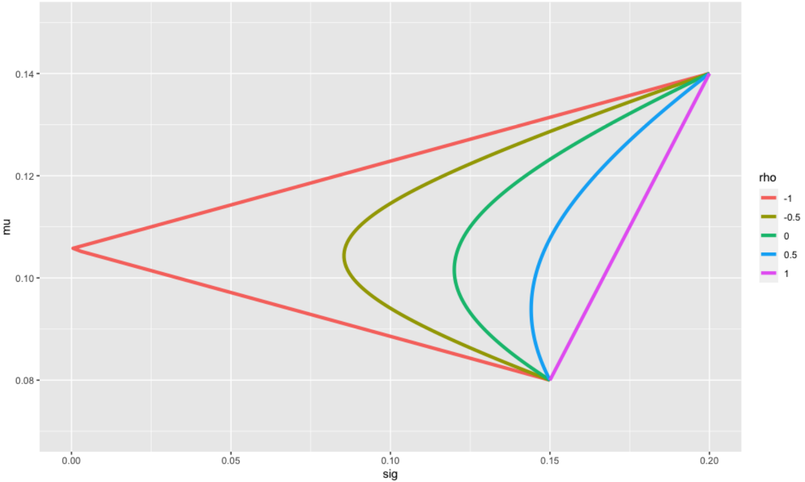

Two risky assets + One risk-free asset

- Given two risky assets, we should choose a *mean-variance efficient portfolio* on the efficient frontier.

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Answer: The one with the highest Sharpe's ratio

$$\frac{\mathbb{E}[R_p] - R_f}{\sigma_p}.$$

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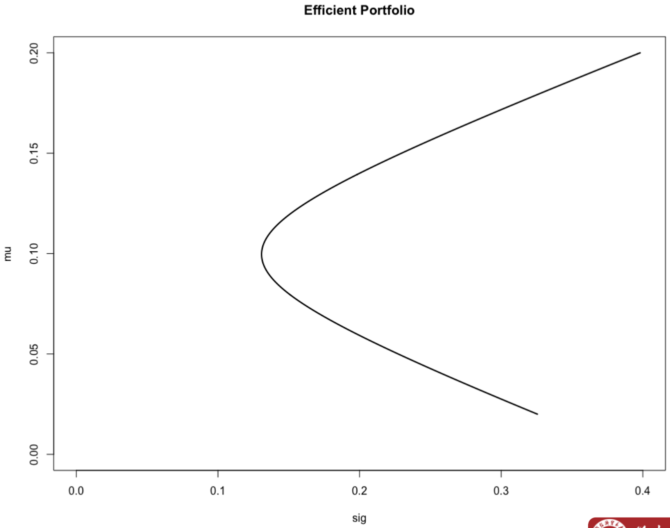

The optimal or efficient portfolios mix the tangency portfolio with the risk-free asset. Each efficient portfolio has two properties:

- it has a higher expected return than any other portfolio with the same or smaller risk, and
- it has a smaller risk than any other portfolio with the same or higher expected return.

Tangency portfolio

Let $V_1 = \mu_1 - R_f$ and $V_2 = \mu_2 - R_f$ be the excess expected returns, then we can show that the tangency portfolio uses weight

$$w_1^* = \frac{V_1\sigma_2^2 - V_2\rho_{12}\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1 + V_2)\rho_{12}\sigma_1\sigma_2}.$$

Mutual fund separation theorem

The efficient portfolios are combinations of the tangency portfolio and the risk-free asset. The expected returns and standard deviation of any efficient portfolio are given by:

$$\mu_p = R_f + w(\mu_{\tan} - R_f)$$
$$\sigma_p = w\sigma_{\tan}$$

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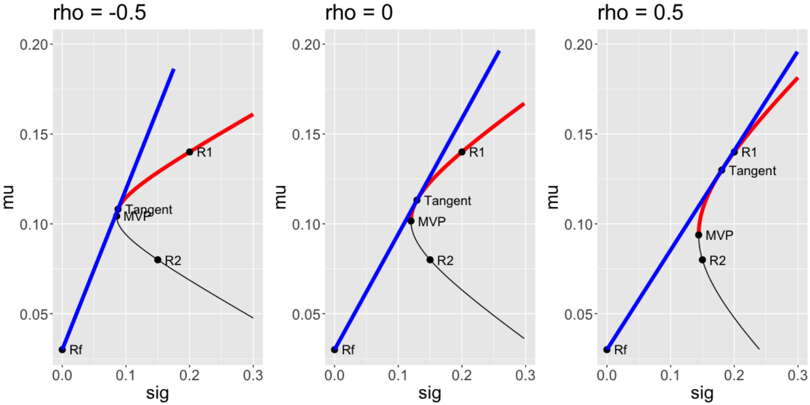

N Risky Assets

N risky assets

Suppose there are N risky assets with returns

$$\mathbf{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}.$$

The expected value and covariance matrix of $\mathbf{R}$ are

$$\boldsymbol{\mu} = \mathbb{E}\left[\mathbf{R}\right] = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} \qquad \boldsymbol{\Sigma} = \operatorname{cov}(\mathbf{R}) = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1}^2 & \dots & \sigma_N^2 \end{pmatrix}.$$

Portfolio return

Let $\mathbf{w}^\intercal = (w_1, \ldots, w_N)$ be the vector of portfolio weights.

1. What condition must $\mathbf{w}$ satisfy?
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Answer:

1. The vector of portfolio weights must satisfy $\mathbf{1}^\intercal \mathbf{w} = \sum_{i=1}^{N} w_i = 1$.
2. The expected value is $\mu_p = \mathbf{w}^\intercal \boldsymbol{\mu} = \sum_{i=1}^{N} w_i \mu_i$. The variance is $\sigma_p^2 = \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w}$.

$N = 2$

The expected return is

$$\mu_p = \mathbf{w}^\top \boldsymbol{\mu} = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = w_1\mu_1 + w_2\mu_2.$$

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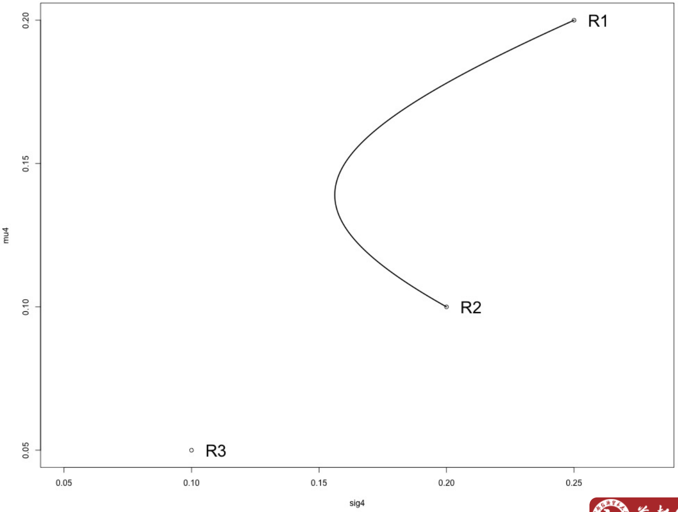
The variance of the portfolio return is

$$\begin{aligned} \sigma_p^2 &= \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\ &= \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2. \end{aligned}$$

$N = 3$



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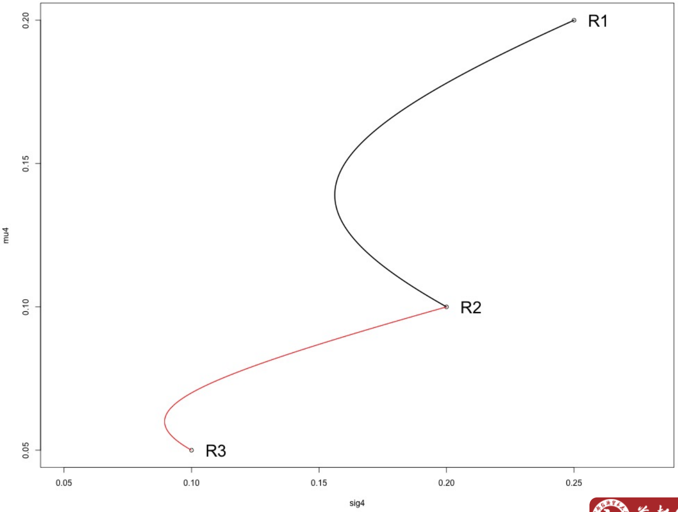

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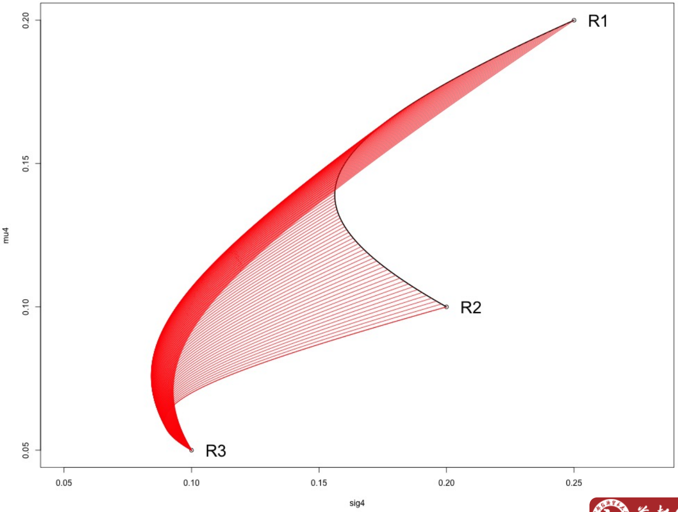

How to find the efficient portfolio?

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- The one with the smallest variance is called the "efficient" portfolio.
- For a given μ_p, an efficient portfolio is one with weights $\mathbf{w}$ such that $\mathbf{w}$ minimizes $\sigma_p^2 = \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w}$, subject to $\mathbf{w}^\intercal \boldsymbol{\mu} = \mu_p$ and $\mathbf{w}^\intercal \mathbf{1} = 1$.

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- The set of efficient portfolios with different target returns form the efficient frontier.

To find the efficient frontier, for each μ_p, we solve the optimization problem

$$\min f(x) := \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$$

subject to

$$\mathbf{1}^\top \mathbf{w} = 1$$
$$\boldsymbol{\mu}^\top \mathbf{w} = \mu_p$$

A Quadratic Programming (QP) problem is an optimization problem with the following form:

$$\min f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^\intercal \mathbf{D}\mathbf{x} - \mathbf{d}^\intercal \mathbf{x}$$

subject to

$$\mathbf{A}_{\mathrm{neq}}^\intercal \mathbf{x} \geq \mathbf{b}_{\mathrm{neq}}$$
$$\mathbf{A}_{\mathrm{eq}}^\intercal \mathbf{x} = \mathbf{b}_{\mathrm{eq}}$$

Our problem of finding an efficient portfolio is a QP problem. The cost function is given by setting

- $\mathbf{x} = \mathbf{w}$
- $\mathbf{D} = 2\boldsymbol{\Sigma}$
- $\mathbf{d} = \mathbf{0}$

The constraint is given by defining

$$\mathbf{A}_{\text{eq}}^{\top} = \begin{pmatrix} \mathbf{1}^{\top} \\ \boldsymbol{\mu}^{\top} \end{pmatrix}, \qquad \mathbf{b}_{\text{eq}} = \begin{pmatrix} 1 \\ \mu_p \end{pmatrix}.$$

Additional constraints can be imposed if needed:

- No-short-sell constraint: $\mathbf{w} \geq \mathbf{0}$.
- To avoid concentration, an investor may wish to constrain the portfolio so that no w_i exceeds a bound λ. In this case, $\mathbf{w} \leq \lambda\mathbf{1}$, or equivalently $-\mathbf{w} \geq -\lambda\mathbf{1}$.

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Combining both, we get

$$\mathbf{A}_{\text{neq}}^{\intercal} = \begin{pmatrix} \mathbf{I} \\ -\mathbf{I} \end{pmatrix}, \qquad \mathbf{b}_{\text{neq}} = \begin{pmatrix} \mathbf{0} \\ -\lambda \mathbf{1} \end{pmatrix}.$$

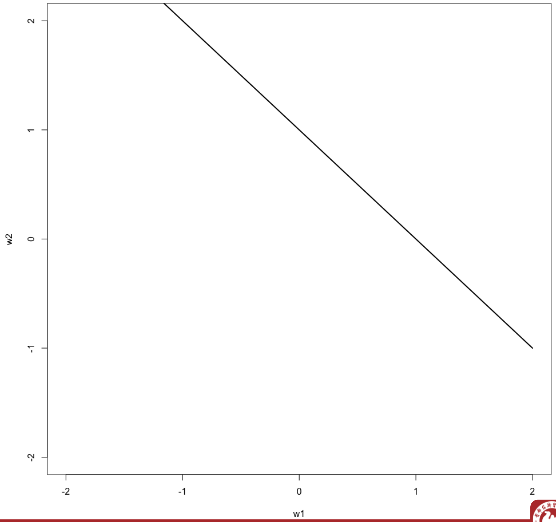
Combining both constraints, the optimization problem becomes

$$\min f(x) := \mathbf{w}^\intercal \mathbf{\Sigma} \mathbf{w}$$

subject to

$$\begin{pmatrix} \mathbf{I} \\ -\mathbf{I} \end{pmatrix} \mathbf{w} \geq \begin{pmatrix} \mathbf{0} \\ -\lambda \mathbf{1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{1}^\intercal \\ \boldsymbol{\mu}^\intercal \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ \mu_p \end{pmatrix}.$$

Quadratic programming



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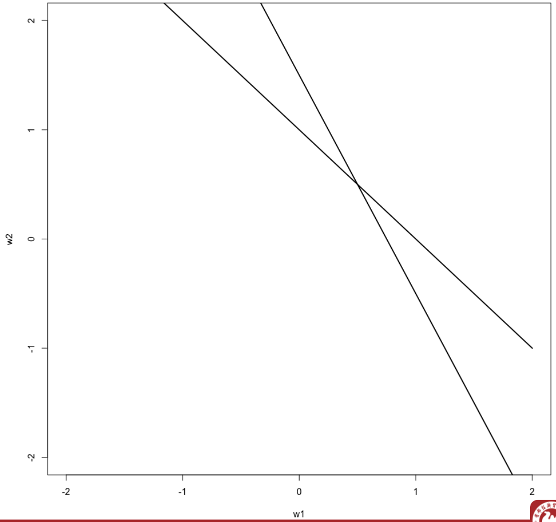

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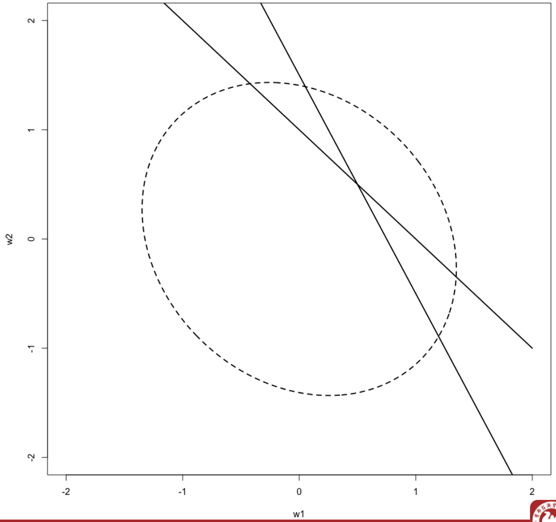

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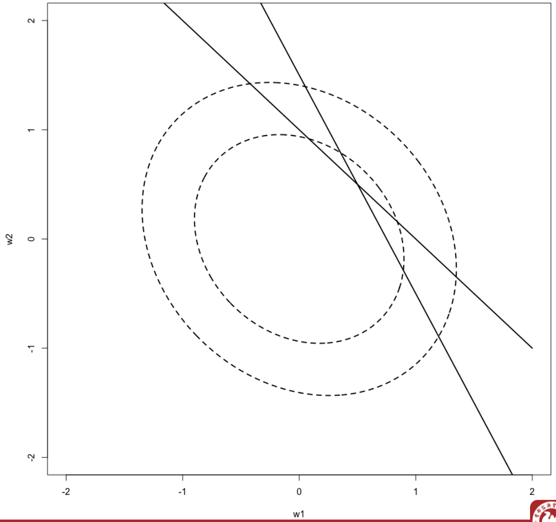

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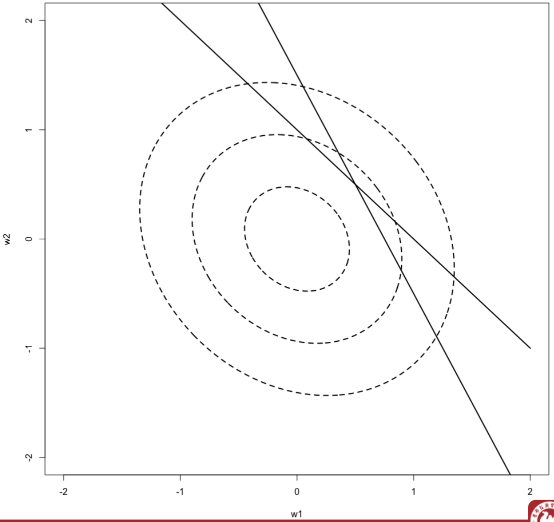

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Answer: The one with the highest Sharpe's ratio

$$\frac{\mathbb{E}\left[R_p\right] - R_f}{\sigma_p}.$$

R Lab

Use the data set `Stock_bond.csv` to answer the following questions:

1. Compute the returns for GM, F, CAT, UTX, MRK and IBM and plot the scatter plots for each stock-pair.
2. Compute their means, standard deviations, covariance matrix and correlation matrix.
3. Suppose you can only invest in GM and one more stock.
   - 3.1 Create the efficient frontier with the two stocks and find the minimum variance portfolio.
   - 3.2 Which firm pair can create an MVP with the smallest variance?

4. Assume the annual risk-free rate is 3%. Find the efficient frontier, the tangency portfolio, and the minimum variance portfolio. What are the Sharpe ratios of the two portfolios?
5. If an investor wants an expected daily return of 0.07%, how should the investor allocate his or her capital to the six stocks and to the risk-free asset?
6. Repeat the above analysis with the following constraints:
   - Short-selling constraint: $w_j \geq 0$ for all j.
   - Short-selling and diversification constraint: $-0.1 \leq w_j \leq 0.5$ for all j.