Financial Modeling and Data Analysis

Supplementary: Matrix and Multivariate Random Variables

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Outline

Vectors and Matrices

Multivariate random variables

Vectors and Matrices

Vector:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Matrix:

$$\mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}.$$

- Zero matrix

$$\boldsymbol{O} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

- Identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

- Addition and subtraction:

$$\mathbf{A} \pm \mathbf{B} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} \pm b_{11} & \dots & a_{1m} \pm b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} \pm b_{n1} & \dots & a_{nm} \pm b_{nm} \end{pmatrix}.$$

- Scalar multiplication:

$$\lambda \mathbf{A} = \begin{pmatrix} \lambda a_{11} & \dots & \lambda a_{1m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n1} & \dots & \lambda a_{nm} \end{pmatrix}.$$

- Properties:
  - $\lambda \mathbf{A} + \lambda \mathbf{B} = \lambda(\mathbf{A} + \mathbf{B})$
  - $\lambda \mathbf{A} + \mu \mathbf{A} = (\lambda + \mu)\mathbf{A}$

- Dot product:

$$\mathbf{a}^\intercal \mathbf{b} = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i.$$

More generally,

$$\mathbf{a}^\intercal \mathbf{S} \mathbf{b} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j s_{ij}.$$

- Matrix multiplication: If $\mathbf{AB} = \mathbf{C}$, then

$$\mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nm} \end{pmatrix}$$

  where

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}.$$

- Properties:
  - $\mathbf{AB} \neq \mathbf{BA}$ in general.
  - $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.
  - $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
  - $\mathbf{AB} = \boldsymbol{O}$ does not imply $\mathbf{A} = \boldsymbol{O}$ or $\mathbf{B} = \boldsymbol{O}$.
  - $\mathbf{AB} = \mathbf{AC}$ does not imply $\mathbf{B} = \mathbf{C}$.

- Transpose of matrix: If $\mathbf{A}^\intercal = \mathbf{B}$, then

$$b_{ij} = a_{ji}.$$

- Properties:
  - $(\mathbf{A}^\intercal)^\intercal = \mathbf{A}$.
  - $(\mathbf{A}+\mathbf{B})^\intercal = \mathbf{A}^\intercal + \mathbf{B}^\intercal$.
  - $(\mathbf{A}\mathbf{B})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$.

- Inverse of matrix:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

- Properties:
  - $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.
  - $(\mathbf{A}^\intercal)^{-1} = (\mathbf{A}^{-1})^\intercal$.
  - If $\mathbf{A}$ is a square matrix, then $(\mathbf{A}^{-1})^n = \mathbf{A}^{-n}$.

Multivariate random variables

Linear functions of random variables

Let $Y = aX + b$, where X is a random variable, while a and b are constants. Then, Let $Y = aX + b$, where X is a random variable, while a and b are constants. Then,

$$\mathbb{E}[Y] = \mathbb{E}[aX + b] = a\mathbb{E}[X] + b = a\mu_X + b.$$

and

$$\operatorname{var}(Y) = \operatorname{var}(aX + b) = a^2 \operatorname{var}(X) = a^2\sigma_X^2.$$

Moreover, $\sigma_Y = |a|\sigma_X$ since $\sigma_Y \geq 0$.

Let $Y = w_1 X_1 + w_2 X_2 = \mathbf{w}^\intercal \mathbf{X}$, where X_1 and X_2 are two random variables. Then, Let $Y = w_1 X_1 + w_2 X_2 = \mathbf{w}^\intercal \mathbf{X}$, where X_1 and X_2 are two random variables. Then,

$$\mathbb{E}\left[Y\right] = \mathbb{E}\left[w_1 X_1 + w_2 X_2\right] = w_1 \mathbb{E}\left[X_1\right] + w_2 \mathbb{E}\left[X_2\right] = w_1 \mu_1 + w_2 \mu_2$$

and

$$\begin{aligned}
\operatorname{var}(Y) &= \operatorname{var}(w_1 X_1 + w_2 X_2) \\
&= \operatorname{var}(w_1 X_1) + \operatorname{var}(w_2 X_2) + 2 \operatorname{cov}(w_1 X_1, w_2 X_2) \\
&= w_1^2 \operatorname{var}(X_1) + w_2^2 \operatorname{var}(X_2) + 2 w_1 w_2 \operatorname{cov}(X_1, X_2) \\
&= w_1^2 \sigma_1 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}.
\end{aligned}$$

In matrix form,

$$\mathbb{E}[Y] = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mathbf{w}^\intercal \boldsymbol{\mu},$$

and

$$\begin{aligned} \operatorname{var}(Y) &= \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \end{pmatrix} \\ &= \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w}. \end{aligned}$$

Let $\mathbf{w} = (w_1, \dots, w_N)^\intercal$ be a vector of scalar and $\mathbf{X} = (X_1, \dots, X_N)^\intercal$ a vector of random variables, where $\boldsymbol{\mu} = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_N])^\intercal$ and

$$\boldsymbol{\Sigma} = \mathbb{E}\left[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^\intercal\right] = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \dots & \sigma_{NN} \end{pmatrix}.$$

Then,

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Then,

$$\mathbb{E}\left[\mathbf{w}^\intercal \mathbf{X}\right] = \mathbf{w}^\intercal \boldsymbol{\mu}, \qquad \mathrm{var}(\mathbf{w}^\intercal \mathbf{X}) = \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w}.$$

Moreover,

$$\mathrm{cov}(\mathbf{w}_1^\intercal \mathbf{X}, \mathbf{w}_2^\intercal \mathbf{X}) = \mathbf{w}_1^\intercal \boldsymbol{\Sigma} \mathbf{w}_2 = \mathbf{w}_2^\intercal \boldsymbol{\Sigma} \mathbf{w}_1.$$