Financial Modeling and Data Analysis

Capital Asset Pricing Model

CHEUNG Ying Lun

Capital University of Economics and Business

Outline

Utility and Portfolio Choice

Utility function

Economists model economic decisions by using a *utility function*, which measures the welfare or satisfaction of a person. Let X be the wealth or consumption of a person, a utility function has the following properties:

1. $U(0) = 0$;
2. U is strictly increasing, $U(X)' > 0$;
3. the first derivative $U(X)'$ is strictly decreasing, $U(X)'' < 0$.

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A rational person will make investment decisions so as to maximize

$$\mathbb{E}[U(X)] = \mathbb{E}[U(X_0(1+R))].$$

Example

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Answer: No. The expected utility of accepting the bet is smaller than that of declining the bet. It shows that the person is *risk averse*.

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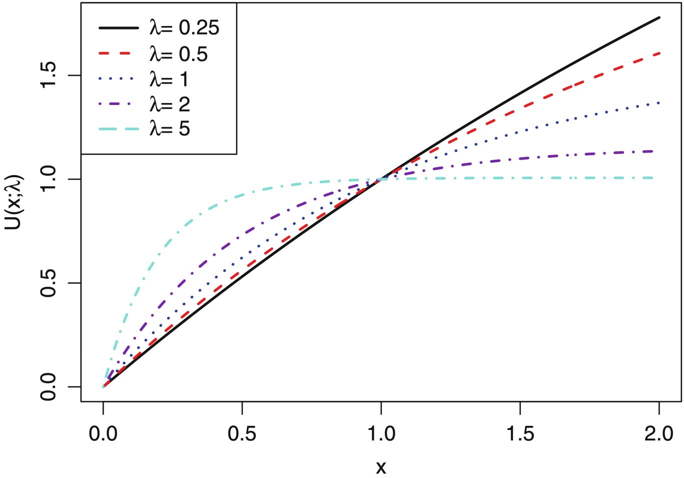
- $U'(x;\lambda) = \lambda \exp(-\lambda x) > 0$;
- $U''(x;\lambda) = -\lambda^2 \exp(-\lambda x) < 0$.

Utility function



Theorem

Suppose the utility function of an investor is given by

$$U(x;\lambda) = 1 - \exp(-\lambda x), \qquad \lambda > 0.$$

Suppose that the initial wealth of the investor is X_0*, and that the returns of the risky assets are jointly normal. Then, the investor will choose a mean-variance efficient portfolio.*

Theorem

If returns on all portfolios are normally distributed, then the portfolio that maximizes expected utility is on the efficient frontier.

Capital Market Line

What would happen if all investors shared an identical investable universe and used the same input list to draw their efficient frontiers?

Assumptions

- Individual behavior
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- Market structure
    1. No taxes or transaction costs.
    2. All assets are publicly held and traded on public exchanges.
    3. Investors can borrow or lend at a common risk-free rate.
    4. Investors can take short positions on traded securities.

Mutual fund separation theorem

The efficient portfolios are combinations of the tangency portfolio and the risk-free asset. The expected returns and standard deviation of any efficient portfolio are given by:

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$$\sigma_p = w\sigma_{\tan}$$

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The efficient frontier, also called the *Capital Allocation Line* in this case, is given by

$$\mu_p = R_f + \frac{\mu_{\tan} - R_f}{\sigma_{\tan}}\sigma_p.$$

Implications

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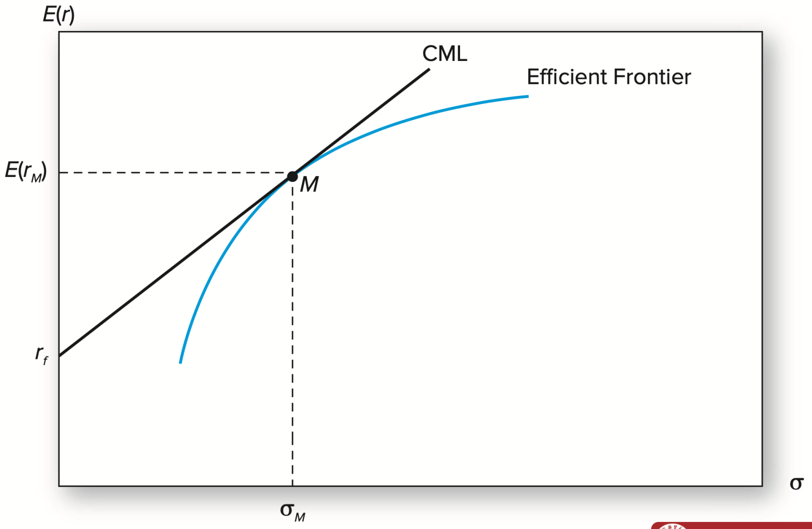
2. Each investor holds a combination of the risk-free asset, and the *same* tangency portfolio.
3. The market portfolio will have the same weights as the tangency portfolio.
4. The capital allocation line based on each investor's optimal risky portfolio will in fact also be the capital *market* line.

Capital Market Line



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Re-writing the CML as

$$\frac{\mu_p - R_f}{\sigma_p} = \frac{\mu_M - R_f}{\sigma_M},$$

the CML says that all efficient portfolios have the same Sharpe's ratio as the market portfolio.

Exercise

Suppose that the risk-free rate of interest is 4% and the expected rate of return on the market portfolio is 14%. The standard deviation of the market portfolio is 12%.

1. According to the CAPM, what is the efficient way to invest with an expected rate of return of 11?%
2. What is the risk (standard deviation) of the portfolio?

Security Market Line

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Answer: The asset's contribution to the risk premium and variance of the market portfolio are respectively $w_j(\mu_j - R_f)$ **and** $w_j \operatorname{cov}(r_j, r_M)$**. The reward-to-risk ratio for investments in this asset is**

$$\frac{w_j(\mu_j - R_f)}{w_j \operatorname{cov}(r_j, r_M)} = \frac{\mu_j - R_f}{\operatorname{cov}(r_j, r_M)}.$$

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- Re-arranging yields the *security market line*

$$\mu_j - R_f = \frac{\text{cov}(r_j, r_M)}{\sigma_M^2}(\mu_M - R_f)$$
$$= \beta_j(\mu_M - R_f).$$

The security market line can be written as

$$\mathbb{E}[R_j] = R_f + \beta_j(\mathbb{E}[R_M] - R_f).$$

The expected return of a risky asset is the sum of two parts:

- Compensation for waiting (Time value of money), and
- Compensation for taking risk, which is the product of:
  - a "benchmark risk premium" (that of the market portfolio), and
  - the relative risk of the particular asset as measured by its beta.

- The risk premium does not depend on the total volatility of an asset.
- The CAPM predicts that systematic risk should "be priced," meaning that it commands a risk premium.
- Firm-specific risk should not be priced by the market, since it is diversifiable.

Example

Consider a pharmaceutical company, which stock performance depends on whether or not a new drug under development is successful. Therefore, the stock return has an extremely high variance. Should investors expect a high return for this reason?

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Answer: NO! Its contribution to overall portfolio risk is low.

Theorem

Let μ_M be the expected return of the market portfolio and σ_M^2 be the variance. Suppose Asset i is one of the risky asset. Then,

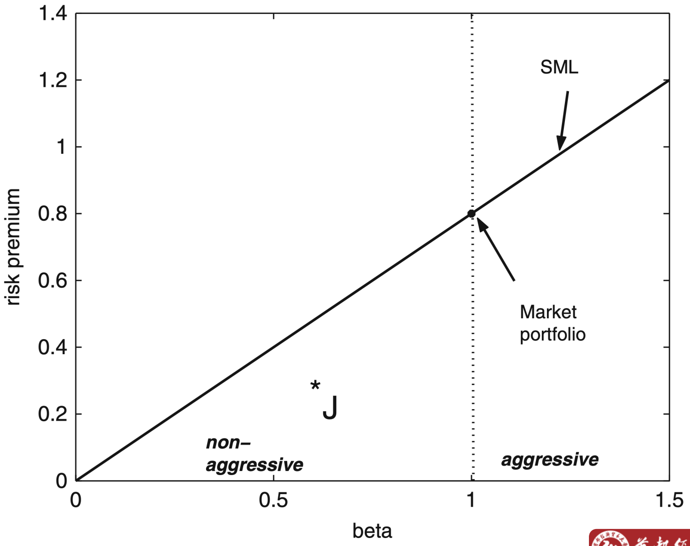
$$\mu_i - R_f = \beta_i(\mu_M - R_f), \qquad \textit{where } \beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}.$$

Here R_f is the risk-free rate, and $\sigma_{i,M} = \operatorname{cov}(r_i, r_M)$.

Exercise

1. Suppose the betas of two risky assets A and B are β_A and β_B respectively. What is the beta of a portfolio with w_A of asset A and $1 - w_A$ of asset B?
2. What is the beta of the market portfolio?

Security Market Line



CML vs SML

Capital market line:

- Risk premiums of *efficient portfolios* against their risks.
- Risk measured by standard deviation.

Security market line:

- Risk premiums of *individual assets* against their risks.
- Risk represented by the contribution of the asset to the portfolio variance, i.e., its beta.

R Lab

Consider a single-index model

$$r_i = \alpha_i + \beta_i r_M + e_i,$$

where r_i is the excess return of Asset i, r_M is the market excess return, α_i is a non-market premium, and e_i a firm-specific surprise with zero mean. Then, taking expectation:

$$\mathbb{E}\left[r_i\right] = \alpha_i + \beta_i \mathbb{E}\left[r_M\right].$$

What does CAPM predict?

The single-index model is a regression model. Given time series $R_{i,t}$, $R_{M,t}$ and $R_{f,t}$ for $t = 1, \ldots, T$, we can calculate $r_{i,t} = R_{i,t} - R_{f,t}$ and $r_{M,t} = R_{M,t} - R_{f,t}$ and estimate the model

$$r_{i,t} = \alpha_i + \beta_i r_{M,t} + e_{i,t}$$

by regressing $r_{i,t}$ on $r_{M,t}$. By testing the null hypothesis that $\alpha_i = 0$, we are testing whether the i-th asset is mispriced according to the CAPM.

Take the S&P 500 index as the market portfolio and the three-month T-bill as the risk free asset and run a single-index model for the stock excess return of Microsoft.

1. Is the estimated alpha *statistically* significant? Is it *economically* significant?
2. Re-run the regression by fixing alpha to be zero. Is there any difference in the result?