Financial Modeling and Data Analysis

Arbitrage Pricing Theory and Factor Models

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Outline

APT: Assumptions
- Factor Models
- Diversification
- Arbitrage

APT: Implications

R Lab

APT: Assumptions

Assumptions

1. Security returns can be described by a factor model.
2. There are sufficient securities to diversify away firm-specific risk.
3. The security market does not allow for the persistence of arbitrage opportunities.

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CAPM as a factor model

The CAPM predicts that

$$\mu_i - R_f = \beta_i(\mu_M - R_f).$$

Let $r_i = R_i - R_f$ and $r_M = R_M - R_f$,

$$r_i = \beta_i r_M + e_i, \qquad \mathbb{E}\left[e_i\right] = 0,$$

where $\beta_i r_M$ represents the *systematic* part, and e_i represents the *firm-specific* part.

CAPM as a factor model

Let $F = r_M$, CAPM can be written as a *single-factor model*

$$r_i = \beta_i F + e_i$$

where

- F is the common risk factor; and
- β_i is the sensitivity of asset i to the factor, also called *factor loadings*.

Factor model

APT postulates that the (excess) return of any asset follows a K-factor model

$$r_i = \mathbb{E}\left[r_i\right] + \beta_{i,1} F_1 + \cdots + \beta_{i,K} F_K + e_i,$$

where

- $\mathbb{E}\left[F_1\right] = \cdots = \mathbb{E}\left[F_K\right] = \mathbb{E}\left[e_i\right] = 0.$
- $\text{cov}(e_i, F_k) = 0$ for all k.
- The risk factors themselves, and the firm-specific shocks for different stocks, may be correlated.

Factor model

Examples of factors include:

1. returns on the market index;
2. growth rate of the GDP;
3. inflation rate or changes in this rate;
4. interest rate spread;
5. return on some portfolio of stocks, for example, all stocks with a high ratio of book equity to market equity;
6. the difference between the returns on two portfolios, for example, stocks with high and low BE/ME values.

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Suppose there is only one factor, and let F be the deviation from its mean. Then, the excess return on an N-asset portfolio is

$$r_p = \mathbb{E}\left[r_p\right] + \beta_p F + e_p$$

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where

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i, \qquad \mathbb{E}[r_p] = \sum_{i=1}^{N} w_i \mathbb{E}[r_i]$$

and

$$e_p = \sum_{i=1}^{N} w_i e_i.$$

Since F and e_i are not correlated,

$$\sigma_p^2 = \beta_p^2 \sigma_F^2 + \sigma^2(e_p),$$

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$$\sigma^2(e_p) = \sum_{i=1}^{N} w_i^2 \sigma_i^2,$$

assuming that the firm-specific e_i are also uncorrelated.

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$$\sigma^2(e_p) = \sum_{i=1}^{N} w_i^2 \sigma_i^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2$$
$$= \frac{1}{N}\left[\frac{1}{N}\sum_{i=1}^{N} \sigma_i^2\right] = \frac{1}{N}\overline{\sigma}^2(e_i).$$

  Therefore, $\sigma^2(e_p) \to 0$ as $N \to \infty$.

- We define a *well-diversified portfolio* as one with $\sigma^2(e_p) \to 0$.

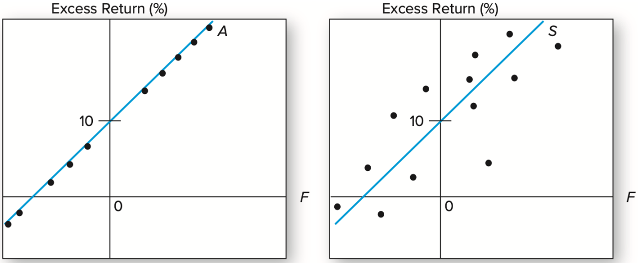
Example

Suppose that the firm-specific risk of each risky asset is finite and uncorrelated with each other. Show that an N-asset portfolio with positive weights $w_i > 0$ is well-diversified if each weight approaches zero as N increases.

Well-diversified portfolios

Since the mean and variance of the firm-specific risk of a well-diversified portfolio are both (close to) zero, any realized value of e_p will be virtually zero. Therefore, we can write

$$r_p = \mathbb{E}\left[r_p\right] + \beta_p F.$$



Assumptions

1. Security returns can be described by a factor model.
2. There are sufficient securities to diversify away firm-specific risk.
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- The *Law of One Price* states that if two assets are equivalent in all economically relevant respects, then they should have the same market price.

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- Enormous profit can be reaped by *long-short positions* if the Law of One Price is violated.
- Market prices will move to rule out arbitrage opportunities.

Arbitrage portfolio

If there is an arbitrage opportunity in the market, an investor can construct an arbitrage portfolio to earn risk-free returns. An arbitrage portfolio needs to satisfy the following conditions:

- *Zero net investment*: $w_1 + \cdots + w_N = 0$.
- *Zero betas*: $w_1\beta_{1,k} + \cdots + w_N\beta_{N,k} = 0$ for all k.
- *Positive returns*: $w_1\mathbb{E}[R_1] + \cdots + w_N\mathbb{E}[R_N] > 0$.

The arbitrage portfolio is usually formed with well-diversified portfolios such that it is completely risk-free.

Example

Suppose the return of any asset in the market follows a one-factor model. Consider the expected returns and betas of the following three well-diversified portfolios:

Portfolio	μ_i (%)	β_i
1	15	0.9
2	21	3.0
3	12	1.8

Construct an arbitrage portfolio.

APT: Implications

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- Their risk premiums must be proportional to beta.
- Since all well-diversified portfolios are perfectly correlated with the factor, if a market index portfolio is well diversified, its return will perfectly reflect the value of the factor.
- For any well-diversified portfolio, the expected excess return must be

$$\mathbb{E}\left[r_p\right] = \beta_p \mathbb{E}\left[r_M\right].$$

Multi-factor APT

- The single-factor model can be generalized to a multi-factor model

$$r_i = \mathbb{E}\left[r_i\right] + \sum_{k=1}^{K} \beta_{i,k} F_k + e_i.$$

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- A well-diversified portfolio constructed to have a beta of 1 on one of the factors and a beta of zero on any other factor is called a *factor portfolio*, or a *tracking portfolio*.
- Let f_k be the excess return on the tracking portfolio of the k-th factor, then

$$\mathbb{E}[r_i] = \sum_{k=1}^{K} \beta_{i,k} \mathbb{E}[f_k].$$

Exercise

Suppose the security returns in a market can be described by a two-factor model. Let r_1 and r_2 be the excess returns of two well-diversified portfolios. Construct the factor portfolios with the two well-diversified portfolios.

Risk-return dominance vs arbitrage

- Risk-return dominance:
  - When an equilibrium price relationship is violated, many investors will make *limited* portfolio changes.
  - Aggregation of a large number of these changes is required to restore equilibrium prices.

Risk-return dominance vs arbitrage

- Risk-return dominance:
  - When an equilibrium price relationship is violated, many investors will make *limited* portfolio changes.
  - Aggregation of a large number of these changes is required to restore equilibrium prices.
- Arbitrage
  - Each investor wants to take as large a position as possible.
  - It will not take many investors to bring about the price pressures necessary to restore equilibrium

APT vs CAPM

Advantages of APT:

- Only a limited number of investors are needed to restore any disequilibrium.
- The assumption that a rational capital market will preclude arbitrage opportunities is plausible.
- A well-diversified index portfolio, instead of the impossible-to-observe market portfolio of all assets, can suffice for the APT.

APT vs CAPM

Disadvantages of APT:

- APT does not imply that the expected return–beta relationship hold for all assets.
- Even large portfolios may have non-negligible residual risk, i.e., not well-diversified.

R Lab

Fitting factor models

There are three ways of fitting factor models:

1. Time series factor models
2. Cross-sectional factor models
3. Statistical factor models

Suppose the excess return of an asset follows a K-factor model

$$r_i = \mathbb{E}[r_i] + \sum_{k=1}^{K} \beta_{i,k} F_k + e_i.$$

Then, let f_k be the excess returns of a factor portfolio, we can estimate a K-index model by regression

$$r_{it} = \beta_{0,i} + \sum_{k=1}^{K} \beta_{i,k} f_{kt} + e_{it}.$$

Fama-French Three-Factor Model

Fama and French have developed a fundamental factor model with three risk factors:

$$r_i = \beta_{0,i} + \beta_{1,i} r_M + \beta_{2,i}\text{SMB} + \beta_{3,i}\text{HML} + e_i$$

where

- $R_M - R_f$: the excess market return
- SMB: the size factor
- HML: the value factor

To estimate the betas, we can run the following regression

$$r_{i,t} = \beta_{0,i} + \beta_{1,i} r_{M,t} + \beta_{2,i}\text{SMB}_t + \beta_{3,i}\text{HML}_t + e_{i,t}$$

where

- $r_{M,t}$: the excess return of the market index
- SMB: the difference in returns on a portfolio of small stocks and a portfolio of large stocks
- HML: the difference in returns on a portfolio of high book-to-market value (BE/ME) stocks and a portfolio of low BE/ME stocks

In matrix form, the vector of N asset returns follows

$$\mathbf{R} = \boldsymbol{\beta}_0 + \mathbf{B}^\intercal \mathbf{F} + \mathbf{e}.$$

Their expectations and covariance matrix are

$$\boldsymbol{\mu} = \boldsymbol{\beta}_0 + \mathbf{B}^\intercal \boldsymbol{\mu}_{\mathbf{F}}, \qquad \boldsymbol{\Sigma} = \mathbf{B}^\intercal \boldsymbol{\Sigma}_{\mathbf{F}} \mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{e}},$$

where

$$\mathbf{B} = \begin{pmatrix} \boldsymbol{\beta}_1 & \ldots & \boldsymbol{\beta}_N \end{pmatrix}$$

and $\boldsymbol{\beta}_i^\intercal = (\beta_{1,i}, \ldots, \beta_{K,i})$.

Cross-sectional factor model

- Many economically relevant variables are not time-series data
- Examples:
  - dividend yields
  - book-to-market ratio
  - industry
- Estimate a cross-sectional factor model:

$$R_i = \beta_0 + \beta_1 \texttt{tech}_i + \beta_2 \texttt{oil}_i + e_i.$$

- Time-series factor model:
  - One estimates parameters, one asset at a time, using multiple holding periods.
  - The factors are directly measured and the loadings are the unknown parameters to be estimated by regression.
- Cross-sectional factor model:
  - One estimates parameters, one single holding period at a time, using multiple assets.
  - The loadings are directly measured and the factor values are estimated by regression.

Statistical factor model

If neither the factor values nor the loadings are directly observable, one can still employ a statistical factor model:

$$\mathbf{R}_t = \boldsymbol{\beta}_0 + \mathbf{B}^\intercal \mathbf{F}_t + \mathbf{e}_t$$

where

$$\boldsymbol{\Sigma}_{\mathbf{R}} = \mathbf{B}^\intercal \boldsymbol{\Sigma}_{\mathbf{F}} \mathbf{B} + \boldsymbol{\Sigma}_{\mathbf{e}}.$$

Only $\mathbf{R}_t$ is observed, and so only $\boldsymbol{\Sigma}_{\mathbf{R}}$ can be directly estimated.

- Identification problem in a statistical factor model: For any invertible matrix $\mathbf{Q}$,

$$\mathbf{B}^\intercal \mathbf{F}_t = \mathbf{B}^\intercal \mathbf{Q}^{-1} \mathbf{Q} \mathbf{F}_t.$$

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  The model is only identifiable up to a nonsingular linear transformation.
- A set of constraints is needed to identify the parameters, e.g.,
  - $\boldsymbol{\Sigma}_{\mathbf{F}} = \mathbf{I}$.
  - $\mathbf{B}\boldsymbol{\Sigma}_{\mathbf{e}}^{-1}\mathbf{B}^\intercal$ is diagonal.
  - *Varimax* rotations: make each loading either small or large by maximizing the sum of the variances of the squared loadings.